

Differential Rigidity of Anosov Actions of Higher Rank Abelian Groups and Algebraic Lattice Actions

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To Dmitry Viktorovich Anosov on his sixtieth birthday

Abstract

We show that most homogeneous Anosov actions of higher rank Abelian groups are locally C^∞ -rigid (up to an automorphism). This result is the main part in the proof of local C^∞ -rigidity for two very different types of algebraic actions of irreducible lattices in higher rank semisimple Lie groups: (i) the Anosov actions by automorphisms of tori and nil-manifolds, and (ii) the actions of cocompact lattices on Furstenberg boundaries, in particular, projective spaces. The main new technical ingredient in the proofs is the use of a proper “non-stationary” generalization of the classical theory of normal forms for local contractions.

1 Introduction

The theory of Anosov systems is one of the crown jewels of dynamics. The notion represents the most perfect kind of global hyperbolic behavior which renders itself to qualitative analysis relatively easily while the global classification remains to a large extent mysterious. The concept was introduced by D.V. Anosov in [1] and the fundamentals of the theory were developed in his classic monograph [2] which during almost thirty years that passed after its publication has been serving as a source of ideas and inspiration for new work. Anosov himself called the class of systems “U-systems” after the first letter of the Russian word “usloviye”, which means simply “condition”, and the name “U-systems” was being used in the Russian language publications for a number of years. However, the term “Anosov systems”, coined by Smale [44], who seems to have immediately recognized both the importance of the notion and the credit due to its author, immediately became current in publications outside of the Soviet Union and eventually was adopted universally.

We are not going to discuss here either the history of the theory of Anosov systems or its general impact in dynamics and beyond its borders. We hope to come back to these topics in another paper. We are concerned here with only one line of development which has been spawned by the seminal work of Anosov. The original notion of Anosov system came in two varieties: Anosov diffeomorphisms (or “cascades” in Anosov’s own terminology), i.e. the actions of the group \mathbb{Z} of integers, and Anosov flows, i.e. actions of the group \mathbb{R} . The former are *structurally stable* as \mathbb{Z} actions: every C^1 diffeomorphism close to an Anosov diffeomorphism is *topologically conjugate* to it. The latter allow countably many moduli of topological conjugacy as \mathbb{R} actions, e.g. the

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lengths of closed orbits; they are nevertheless structurally stable in the orbit equivalence sense: the orbit foliation of an Anosov flow is topologically conjugate to the orbit foliation of any C^1 flow sufficiently close to it in the C^1 topology. The development which concerns us here deals with the notion of an *Anosov action* of a group (either discrete or continuous), more general than \mathbb{Z} or \mathbb{R} . This concept was originally introduced by Pugh and Shub [33] in the early seventies (their notion coincides with ours for abelian groups and is different in general) and for many years fundamental differences between the “classical” (\mathbb{Z} and \mathbb{R}) and other cases went mostly unnoticed, except for the one-dimensional situation [42]. Only with the progress of the “geometric rigidity” approach to the actions of higher-rank semisimple Lie groups and lattices in such groups initiated by Zimmer in the mid-eighties did these differences come to the attention of researchers beginning with [18]. In particular, it turned out that both local *differentiable* stability (or rigidity) and cocycle rigidity are present and in fact typical already for Anosov actions of higher-rank abelian groups, i.e. \mathbb{Z}^k and \mathbb{R}^k for $k \geq 2$.

In this paper we bring the study of the local differentiable rigidity of algebraic Anosov actions of \mathbb{Z}^k and \mathbb{R}^k on compact manifolds as well as orbit foliations of such actions, started in [18, 21, 22, 24], to a near conclusion. Our results for the abelian group actions formulated in Section 2.1 allow us to obtain comprehensive local C^∞ rigidity for two very different types of algebraic actions of irreducible lattices in higher rank semisimple Lie groups: (i) the Anosov actions by automorphisms of tori and nilmanifolds (Theorem 15, Section 3), and (ii) the actions on Furstenberg boundaries, in particular projective spaces (Theorem 17, Section 4). While the latter area was virtually unapproachable save for a special result by Kanai [13], the former was extensively studied beginning from [11, 18] and continuing in [20, 34, 37, 38, 19, 35, 36]. Our result (Theorem 15 in Section 3) substantially extends all these earlier works and brings the question to a final solution. It also gives further credence to the global conjecture that all Anosov actions of such lattices are smoothly equivalent to one of the above. Such a classification was established for the much more special class of Cartan actions in [8].

The objects considered in this paper, i.e. group actions and foliations, are assumed to be of class C^∞ . Accordingly the basic notions, including the structural stability are adapted to this case while they sometimes appear more naturally for the lower regularity. In fact, for each of our results a certain finite degree of regularity is sufficient resulting in a loss of regularity in the conjugating diffeomorphisms. We leave the detailed study of optimal regularity conditions to a later paper. In proper places we make specific comments about the degree of regularity sufficient to guarantee existence of C^1 conjugating diffeomorphisms.

Let M be a compact manifold. We call two foliations \mathcal{F} and \mathcal{G} on M *orbit equivalent* if there is a homeomorphism $\phi : M \rightarrow M$ such that ϕ takes the leaves of \mathcal{F} to those of \mathcal{G} . We call ϕ an *orbit equivalence*. If ϕ is a C^∞ -diffeomorphism, we call \mathcal{F} and \mathcal{G} C^∞ -*orbit equivalent*.

Endow the space of foliations on M with the C^1 -topology, i.e. two foliations are close if their tangent distributions are C^1 -close. We call a C^∞ -foliation \mathcal{F} *structurally stable* if there is a neighborhood U of \mathcal{F} such that any foliation in U is orbit equivalent to \mathcal{F} . We call a C^∞ -foliation \mathcal{F} C^∞ -*locally rigid* if there is a neighborhood U of \mathcal{F} such that any C^∞ foliation in U is C^∞ -orbit equivalent to \mathcal{F} .

Call a C^∞ -action of a finitely generated discrete group Γ *structurally stable* if any C^∞ perturbation of the action which is C^1 -close on a finite generating set is conjugate to it via a homeomorphism. Call such a Γ -action *locally C^∞ -rigid* if any perturbation of the action which is C^1 -close on a finite generating set is conjugate to the original action via a C^∞ diffeomorphism. We say that two actions of a group G *agree up to an automorphism* if the second action can be obtained from the first by composition with an automorphism of the underlying group. Call a C^∞ -action

of a Lie group G *locally C^∞ -rigid* if any perturbation of the action which is C^1 -close on a compact generating set is C^∞ -conjugate up to an automorphism.

The notions of C^r , $r \geq 1$ local rigidity for foliations and actions are defined accordingly.

For $\Gamma = \mathbb{Z}$ or \mathbb{R} , i.e. for the classical dynamical systems, diffeomorphisms (\mathbb{Z} -actions) and flows (\mathbb{R} -actions), local C^∞ -rigidity never takes place. Moreover, the orbit foliations of flows are not locally C^∞ -rigid either. In those cases, it does not help to allow the conjugacy in the definitions to be only a C^1 diffeomorphism. Structural stability on the other hand is a rather wide-spread (although not completely understood) phenomenon. See [17] for a general background and [41, 28] for the definitive results on structural stability in C^1 (but not C^r , $r \geq 2$) topology. Since local C^∞ -rigidity implies structural stability one can immediately see from the necessary conditions for structural stability that there are always moduli of C^1 conjugacy in the structurally stable case thus showing impossibility of differentiable rigidity.

Certain structural stability results, namely those by Hirsch, Pugh and Shub [9], are important for our purposes. They established structural stability of central foliations of certain partially hyperbolic dynamical systems which implies in particular that the orbit foliations of hyperbolic (Anosov) actions of \mathbb{R}^k are structurally stable. At the beginning of Section 2.2 we explain how this fact is used as the starting point in our proof of local differentiable rigidity.

Contrary to the classical case, in the higher-rank situation of a \mathbb{Z}^k or \mathbb{R}^k -action for $k \geq 2$, C^∞ local rigidity is possible and in fact seems to be as closely related to the hyperbolic behavior as structural stability is in the classical case. Existence of this phenomenon was first discovered in [18, Theorem 4.2] for certain (in a sense “maximal”) actions of \mathbb{Z}^k by toral automorphisms and was extended in [24] to a broader class of Anosov actions by both \mathbb{R}^k and \mathbb{Z}^k (multiplicity-free standard actions). The results on cocycle rigidity which first appeared in [24] are fairly definitive and they appeared in print as [21, 22]. On the other hand, the local rigidity result for actions in [24] was still too restrictive; in particular, it covered the key situation of the Weyl chamber flows only for split semi-simple groups. In the present paper we obtain much more general rigidity results which are close to being definitive. We prove C^∞ local rigidity of most known irreducible Anosov actions of \mathbb{Z}^k and \mathbb{R}^k as well as the orbit foliations of the latter. (Theorem 1, Corollaries 2 through 5). All such actions are essentially algebraic (see Section 2.1 below) and the remaining technical assumption of semisimplicity of the linear part of the action is satisfied in all interesting cases including the Weyl chamber flows, which appear in applications to the rigidity of actions of irreducible cocompact lattices in higher rank semisimple Lie groups on Furstenberg boundaries (Section 4), and those actions by automorphisms of tori and nil-manifolds which are needed for the proof of rigidity of actions of irreducible lattices in higher rank semisimple Lie groups by such automorphisms (Section 3).

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The principal results of this paper are announced in [25].

2 Anosov actions of higher rank Abelian groups

2.1 The main results

We will consider “essentially algebraic” Anosov actions of either \mathbb{R}^k or \mathbb{Z}^k . We recall that an action of a group G on a compact manifold is *Anosov* if some element $g \in G$ acts normally hyperbolically with respect to the orbit foliation (cf. [21] for more details). To clarify the notion of an algebraic action, let us first define affine algebraic actions of discrete groups. Let H be a connected Lie group with $\Lambda \subset H$ a cocompact lattice. Define $\text{Aff}(H)$ as the set of diffeomorphisms of H which map right invariant vectorfields on H to right invariant vectorfields. Define $\text{Aff}(H/\Lambda)$ to be the diffeomorphisms of H/Λ which lift to elements of $\text{Aff}(H)$. Finally, define an action ρ of a discrete group G on H/Λ to be *affine algebraic* if $\rho(g)$ is given by some homomorphism $G \rightarrow \text{Aff}(H/\Lambda)$. Let \mathfrak{h} be the Lie algebra of H . Identifying \mathfrak{h} with the right invariant vectorfields on H , any affine algebraic action determines a homomorphism $\sigma : G \rightarrow \text{Aut } \mathfrak{h}$. Call σ the *linear part* of this action. We will also allow quotient actions of these on finite quotients of H/Λ , e.g. on infranilmanifolds. For any Anosov algebraic action of a discrete group G , H has to be nilpotent (cf. eg. [8, Proposition 3.13]).

We define algebraic \mathbb{R}^k -actions as follows. Suppose $\mathbb{R}^k \subset H$ is a subgroup of a connected Lie group H . Let \mathbb{R}^k act on a compact quotient H/Λ by left translations where Λ is a lattice in H . Suppose C is a compact subgroup of H which commutes with \mathbb{R}^k . Then the \mathbb{R}^k -action on H/Λ descends to an action on $C \backslash H/\Lambda$. The general algebraic \mathbb{R}^k -action ρ is a finite factor of such an action. Let \mathfrak{c} be the Lie algebra of C . The *linear part* of ρ is the representation of \mathbb{R}^k on $\mathfrak{c} \backslash \mathfrak{h}$ induced by the adjoint representation of \mathbb{R}^k on the Lie algebra \mathfrak{h} of H .

Let us note that the suspension of an algebraic \mathbb{Z}^k -action is an algebraic \mathbb{R}^k -action (cf. [21, 2.2]).

The next theorem is our principal technical result for algebraic Anosov \mathbb{R}^k -actions.

We denote the strong stable foliation of $a \in \mathbb{R}^k$ by \mathcal{W}_a^- , the strong stable distribution by E_a^- , and the 0-Lyapunov space by E_a^0 . Note that E_a^0 is always integrable for algebraic actions. Denote the corresponding foliation by \mathcal{W}_a^0 . Also note that any algebraic action leaves a Haar measure μ on the quotient invariant. Given a collection of subspaces of a vectorspace, call a nontrivial intersection *maximal* if it does not contain any other nontrivial intersection of these subspaces.

Theorem 1 *Let ρ be an algebraic Anosov action of \mathbb{R}^k , for $k \geq 2$, such that the linear part of ρ is semisimple. Assume that for any maximal nontrivial intersection $\bigcap_{i=1\dots r} \mathcal{W}_{b_i}^-$ of stable manifolds of elements $b_1, \dots, b_r \in \mathbb{R}^k$ there is an element $a \in \mathbb{R}^k$ such that for a.e. $x \in M$, $\bigcap_{i=1\dots r} E_{b_i}^-(x) \subset E_a^0(x)$ and such that a.e. leaf of the intersection $\bigcap_{i=1\dots r} \mathcal{W}_{b_i}^-$ is contained in an ergodic component of the one-parameter subgroup $t a$ of \mathbb{R}^k (w.r.t. Haar measure).*

Then the orbit foliation of ρ is locally C^∞ -rigid. In fact, the orbit equivalence can be chosen C^1 -close to the identity. Moreover, the orbit equivalence is transversally unique, i.e. for any two different orbit equivalences close to the identity, the induced maps on the set of leaves agree.

The technical condition in this theorem guarantees that certain nontrivial intersections of stable manifolds are contained in the the closure of the orbit of a generic point x under one-parameter groups which translate these intersections by isometries. It is similar to the condition of the main technical theorem of [23], Theorem 5.1, on invariant measures for algebraic Anosov \mathbb{R}^k -actions.

Corollary 2 *Let ρ be an algebraic Anosov action of \mathbb{R}^k , for $k \geq 2$, such that the linear part of ρ is semisimple. Assume that every one-parameter subgroup of \mathbb{R}^k acts ergodically with respect to the*

Haar measure μ . Then the orbit foliation of ρ is locally C^∞ -rigid. In fact, the orbit equivalence can be chosen C^1 -close to the identity. Moreover, the orbit equivalence is transversally unique.

The assumption that any one-parameter subgroup of \mathbb{R}^k acts ergodically is equivalent to the \mathbb{R}^k -action being weakly mixing, as one easily sees. We refer to Brezin and Moore for a more extensive discussion of the ergodicity of homogeneous flows [4]. This condition is automatically satisfied if H is semisimple without compact factors and Λ is an irreducible lattice, as follows from Mautner's theorem [50]. If H is semisimple, then the linear part of the action is automatically semisimple. The action of a split Cartan subgroup A of H on $K \backslash H / \Lambda$ where $\Lambda \subset H$ is a cocompact lattice in H , and K is the compact part of the centralizer of A in H is always Anosov. This is the so-called *Weyl chamber flow* [21]. Hence the assumptions of Corollary 2 are satisfied for the Weyl chamber flows.

More generally, we will establish C^∞ -local rigidity of the twisted Weyl chamber flows introduced in [21]. Let us briefly recall their construction. Let A be a split Cartan subgroup of a real semisimple Lie group G without compact factors, and let K be the compact part of the centralizer of A in G . Let Γ be a cocompact irreducible lattice in G . Suppose that Γ acts on a nilmanifold $M = N / \Lambda$ such that its linearization extends to a homomorphism of G . Let H be the semidirect product $H = G \ltimes N$. Then $\Gamma \ltimes \Lambda$ is a lattice in H . Then A acts on $K \backslash H / \Gamma \ltimes \Lambda$. This action is Anosov provided that the action of Γ on M contains an Anosov automorphism. In this case, we call the A action a *twisted Weyl chamber flow*.

Our conclusions for the Weyl chamber flows and twisted Weyl chamber flows may be summarized as follows.

Corollary 3 *The orbit foliation of the Weyl chamber flow on $K \backslash H / \Lambda$, where H is a semisimple Lie group of rank greater than one without compact factors and $\Lambda \subset H$ is an irreducible lattice, is locally C^∞ -rigid.*

The orbit foliation of a twisted Weyl chamber flows is locally C^∞ -rigid if the acting Cartan subgroup has real rank at least 2.

In the case of Weyl chamber flows also any C^6 perturbation of the orbit foliation in C^1 topology is C^1 conjugate to the orbit foliation of the Weyl chamber flow. For those groups where there are no positively proportional roots restricted to a split Cartan (e.g. $SL(n, \mathbb{R})$, $n \geq 3$ or $SO(p, q)$, $p, q \geq 2$) C^6 in this statement may be replaced by C^4 . See Sections 2.2.1 and 2.2.3, Steps 4 and 5 for explanations.

To get rigidity results for algebraic \mathbb{Z}^k -Anosov actions with linear semisimple part, we can pass to the suspension. Since the suspension of a \mathbb{Z}^k action is never weakly mixing we cannot apply Corollary 2 and have to appeal to Theorem 1 in its full generality. As we will see, the hypothesis of this theorem is guaranteed if all nontrivial elements of the \mathbb{Z}^k -action are weak mixing. By a theorem of W. Parry, an affine automorphism of a nilmanifold H / Λ is weakly mixing precisely when the quotient of the linear part on the abelianization $\frac{\mathfrak{h}}{[\mathfrak{h}, \mathfrak{h}]}$ does not have roots of unity as eigenvalues [31]. As an orbit equivalence close to the identity between suspensions of \mathbb{Z}^k -actions automatically produces a conjugacy between the actions themselves, we get the following corollary.

Corollary 4 *Let ρ be an algebraic Anosov action of \mathbb{Z}^k , $k \geq 2$, on an infranilmanifold with semisimple linear part. Suppose that no nontrivial element of the group has roots of unity as eigenvalues in the induced representation on the abelianization. Then ρ is C^∞ -locally rigid. Moreover, the conjugacy can be chosen C^1 -close to the identity, and is unique amongst conjugacies close to the identity.*

All known Anosov \mathbb{R}^k -actions satisfying the assumptions of Theorem 1 belong to and almost exhaust the list of *standard* \mathbb{R}^k -actions, introduced in [21]. They essentially consist of actions by infranilmanifold (in particular, toral) automorphisms, Weyl chamber flows, twisted Weyl chamber flows and some further extensions (cf. [21] for more details).

For the standard Anosov actions, we showed that every smooth cocycle is smoothly cohomologous to a constant cocycle [21, Theorem 2.9]. As a consequence, all smooth time changes are smoothly conjugate to the original action (possibly composed with an automorphism of \mathbb{R}^k). Combining this with the above results about the C^∞ -rigidity of the orbit foliations, we obtain the following corollary.

Corollary 5 *The standard algebraic Anosov actions of \mathbb{R}^k for $k \geq 2$ with semisimple linear part are locally C^∞ -rigid. Moreover, the C^∞ -conjugacy ϕ between the action composed with an automorphism ρ and a perturbation can be chosen C^1 -close to the identity. The automorphism ρ is unique and also close to the identity. Finally, ϕ is unique amongst conjugacies close to the identity modulo translations in the acting group.*

In the case of Weyl chamber flows, there are in fact very few conjugacies, close to the identity or not. We pursue this in Section 2.5.

For our application to the local rigidity of projective lattice actions, we will need a rigidity result similar to Corollary 3 for Anosov actions of certain reductive groups. Let G be a connected semisimple Lie group with finite center and without compact factors. Let $\Gamma \subset G$ be an irreducible cocompact lattice. Let P be a parabolic subgroup of G , and let H be its Levi subgroup. Thus $P = HU^+$ where U^+ is the unipotent radical of P . (We refer to [48] for details on parabolic subgroups and boundaries of G .) Then H acts on G/Γ by left translations. These actions are Anosov (cf. eg. [26]).

Theorem 6 *If the real rank of G is at least 2, then the orbit foliation \mathcal{O} of H is C^∞ -locally rigid. Moreover, the orbit equivalence can be chosen C^1 -close to the identity.*

We will actually use the following corollary of the proof of the theorem. Let $P = LCU^+$ be the Langlands decomposition of P (with respect to some Iwasawa decomposition $G = KAN$ of G), and let M_P be the centralizer of C in a K . Then the orbit foliation of H on $\Gamma \backslash G$ descends to a foliation \mathcal{R} on $\Gamma \backslash G/M_P$.

Corollary 7 *If the real rank of G is at least 2, then the foliation \mathcal{R} on $\Gamma \backslash G/M_P$ is C^∞ -locally rigid. Moreover, the orbit equivalence can be chosen C^1 -close to the identity.*

2.2 Proof of Theorem 1

The general outline of the proof is as follows. The weak stable foliations of various elements of our action are canonically perturbed to foliations with smooth leaves saturated by the perturbed foliation \mathcal{R} . Due to the Hirsch-Pugh-Shub structural stability theorem for normally hyperbolic systems, there is a Hölder orbit equivalence between \mathcal{O} and \mathcal{R} . This orbit equivalence is smooth along \mathcal{O} , and carries the weak stable foliations of various elements into their canonical perturbations. By transversality, the same is true for the intersections of the weak stable foliations. The tangent bundles of suitable intersections in fact provide a transversal splitting of the tangent bundle. The arguments outlined so far are fairly standard. In particular, they form a basis of our earlier results [24] including the proof of local differential rigidity in the multiplicity free case as well as partial results in more general cases.

The key part in the proof is the smoothness of the orbit equivalence along these intersections. The essential new technique used here is a suitable non-stationary generalization of the classical theory of normal forms for local differentiable contractions [15]. This technique is summarized in the following section. Once we have smoothness of the orbit equivalence along the foliations of the splitting, standard elliptic theory shows smoothness. Theorem 6 dealing with a more general reductive case is proved very similarly so there is no need to repeat all the details. We outline the necessary modifications after the proof of Theorem 1.

2.2.1 Preliminaries on normal forms

We now give a summary of one possible non-stationary generalization of the classical normal form theory for local contractions whose origins go back to Poincaré and which, in its modern form for the C^∞ case, was developed by Sternberg and Chen [45, 6]. It is quite possible that one can find results very similar to the ones below in the vast literature on normal forms. The first author wrote the precise versions needed for our application in [15].

Consider a continuous extension \mathcal{F} of a homeomorphism f on a compact connected metric space X to a neighborhood of the zero section of a vector bundle V which is smooth along the fibers and preserves the zero section. Let $F = D\mathcal{F}_0$ where the derivative is taken at the zero section in the fiber direction. Consider the induced operator F^* on the Banach space of continuous sections of the bundle V endowed with the uniform norm given by $F^*v(x) = F(v(f^{-1}(x)))$. The *characteristic set* of F^* is the set of logarithms of the absolute values of the spectrum of F^* . It consists of the union of finitely many intervals. For notational convenience we will consider a slightly more general situation. Namely, consider a finite set of disjoint intervals $\Delta_i = [\lambda_i, \mu_i]$ on the negative half-line which *contain* the characteristic set. We order the intervals in increasing order such that $\lambda_{i+1} > \mu_i$. Then the bundle V splits into the direct sum of F -invariant subbundles V_1, \dots, V_l such that the spectrum of the restriction $F|_{V_i} \subset \Delta_i$, $i = 1, \dots, l$ [16]. Let m_i be the dimension of the sub-bundle V_i . Call the extension \mathcal{F} a *contraction* if $F = D\mathcal{F}_0$ is a contraction with respect to a continuous family of Riemannian metrics in the fibers. This is equivalent to the condition that the log of the spectral radius of F^* is negative, i.e. $\mu_l < 0$.

We will assume that F^* has *narrow band spectrum*, i.e. that

$$\mu_i + \mu_l < \lambda_i$$

for all $i = 1, \dots, l$ [15].

Represent the space \mathbb{R}^m as the direct sum of subspaces V_1, \dots, V_l of dimension m_1, \dots, m_l correspondingly and let (t_1, \dots, t_l) be the corresponding coordinate representation of a vector $t \in \mathbb{R}^m$. Let $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$; $(t_1, \dots, t_l) \mapsto (P_1(t_1, \dots, t_l), \dots, P_l(t_1, \dots, t_l))$ be a polynomial map preserving the origin. We will say that the map P is of *sub-resonance type* if it has non-zero homogeneous terms in $P_i(t_1, \dots, t_l)$ with degree of homogeneity s_j in the coordinates of t_j , $i = 1, \dots, l$ only if

$$\lambda_i \leq \sum s_j \mu_j \quad (*)$$

Let us notice that the notions of a homogeneous polynomial and degree of homogeneity in a Euclidean space are invariant under linear transformations. Thus the notion of a map of sub-resonance type depends only on the decomposition $\mathbb{R}^m = \bigoplus_{i=1}^l V_i$, but not on a choice of coordinates in each component of this decomposition. Furthermore, if V is a continuous vector bundle which splits into the direct sum of continuous subbundles V_1, \dots, V_m , and if the vectors λ and μ

are given one can define in a natural way the notion of a continuous polynomial bundle map of sub-resonance type.

We will call any inequality of type $(*)$ a *sub-resonance relation*. There are always sub-resonance relations of the form $\lambda_i \leq \mu_j$ for $j = i, \dots, l$. They correspond to the linear terms of the polynomial. We will call such relations *trivial*. The narrow band condition guarantees that for any non-trivial sub-resonance relation $s_j = 0$ for $j = 1, \dots, i$.

Although the composition of maps of sub-resonance type may not be a map of sub-resonance type it is not difficult to see that maps of sub-resonance type with invertible derivative at the origin generate a finite-dimensional group all of whose elements are polynomial maps of degree bounded by a constant which depends only on the vectors λ and μ . We will denote this group by $SR_{\lambda, \mu}$ and will call its elements *sub-resonance generated* polynomial maps. See [15] for more explanations.

In certain cases the maps of sub-resonance type with invertible derivative at the origin already form a group. In particular, if the whole spectrum is sufficiently narrow i.e if $\lambda_1 > 2\mu_l$, then there are no non-trivial sub-resonance relations and hence $SR_{\lambda, \mu}$ is a subgroup of $GL(m, \mathbb{R})$ the group of linear automorphisms of \mathbb{R}^m . The next simplest case appears in the perturbation of the point spectrum with 2 : 1 resonance. In this case $l = 2$ and $\lambda_1 > \max\{3\mu_2, \mu_1 + \mu_2\}$. The only possible non-trivial sub-resonance relation is $\lambda_1 \leq 2\mu_2$. In this case the group $SR_{\lambda, \mu}$ consist of quadratic maps of the form

$$P(t_1, t_2) = (L_1 t_1 + Q(t_2, t_2), L_2 t_2),$$

where L_1 and L_2 are linear maps and Q is a quadratic form. The above two cases are the only ones which appear in the consideration of small perturbations of Weyl chamber flows since only the double of a root (restricted to a split Cartan) can be a (restricted) root.

Call two extensions *conjugate* if there exists a continuous family of local C^∞ diffeomorphisms of the fibers $V(x)$, preserving the origin which transforms one extension into the other. The following two theorems on normal forms and centralizers are proved in [15].

Theorem 8 (sub-resonance normal form) *Suppose the extension \mathcal{F} is a contraction and the linear extension $D\mathcal{F}$, has a narrow band spectrum determined by the vectors $\lambda = (\lambda_1, \dots, \lambda_l)$ and $\mu = (\mu_1, \dots, \mu_l)$. There exists an extension $\tilde{\mathcal{F}}$ conjugate to \mathcal{F} such that for every $x \in X$,*

$$\tilde{\mathcal{F}}|_{V(x)}: \bigoplus_{i=1}^l V_i(x) \rightarrow \bigoplus_{i=1}^l V_i(f(x))$$

is given by a polynomial map of sub-resonance type.

Theorem 9 (Centralizer for sub-resonance maps) *Suppose g is a homeomorphism of the space X commuting with f and $\tilde{\mathcal{G}}$ is an extension of g by C^∞ -diffeomorphisms of the fibers (not necessarily a contraction) commuting with an extension $\tilde{\mathcal{F}}$ satisfying the assertion of Theorem 8. Then $\tilde{\mathcal{G}}$ has a similar form:*

$$\tilde{\mathcal{G}}|_{V(x)}: \bigoplus_{i=1}^l V_i(x) \rightarrow \bigoplus_{i=1}^l V_i(g(x))$$

is a polynomial sub-resonance generated map, i.e. an element of the group $SR_{\lambda, \mu}$.

Combining these two theorems we see that a local action of an abelian group by extensions which contains a contraction which has narrow band spectrum can be simultaneously brought to a normal form.

Corollary 10 *Let ρ be a continuous action of \mathbb{R}^k on a compact connected metric space X . Let V be a vector bundle over X . Suppose that σ is a local action of \mathbb{R}^k in a neighborhood of the zero section of V such that σ covers ρ , σ is C^∞ along the leaves and each $\sigma(a)|_{V_x}$ depends continuously on the base point x in the C^∞ -topology. Suppose further that some $a \in \mathbb{R}^k$, $\sigma(a)$ is a contraction and that the induced linear operator on C^0 -sections of V has narrow band spectrum. Then there exist C^∞ changes of coordinates in the fibers V_x , depending continuously on x such that for all $b \in \mathbb{R}^k$, $\sigma(b)|_{V(x)}$ is a polynomial map of sub-resonance type.*

All these results have counterparts for extensions of finite differentiability (M.Guysinsky; in preparation). The precise degree of differentiability both for the normal form and for the centralizer theorems depends on the vectors λ and μ . Let us mention the following specific cases which appear in the perturbations of the Weyl chamber flows (see comments after the formulation of Corollary 3) and hence in the applications to the projective lattice actions on the Furstenberg boundaries (See Section 4). We will refer to the differentiability of an extension meaning a leafwise differentiability continuously changing in the corresponding topology and to differentiability of a normal form meaning the leafwise differentiability of the map bringing an extension to the normal form.

In the narrow spectrum case $\lambda_1 > 2\mu_l$ there is a C^2 linear normal form for a C^4 extension and the C^2 centralizer of a linear extension with this kind of spectrum consist of linear maps. In the case $l = 2$ and $\lambda_1 > \max\{3\mu_2, \mu_1 + \mu_2\}$ there is a C^3 sub-resonance normal form for a C^6 extension and any commuting C^3 extension has the same sub-resonance form.

2.2.2 Coarse Lyapunov decomposition for algebraic actions

Consider an algebraic \mathbb{R}^k -action ρ with linear part σ . We will assume that σ is semisimple. Define the *Lyapunov exponents* of ρ as the log's of the absolute values of the eigenvalues of σ . We get linear functionals $\chi : \mathbb{R}^k \rightarrow \mathbb{R}$. These functionals coincide with the Lyapunov exponents for the Haar measure [21, 15] or for any invariant measure for that matter. Note that ρ is Anosov precisely if the only 0 Lyapunov exponents come from the orbit. There is a splitting of the tangent bundle into \mathbb{R}^k -invariant subbundles $TM = \bigoplus_\chi E_\chi$ such that the Lyapunov exponent of $v \in E_\chi$ with respect to $\rho(a)$ is given by $\chi(a)$. We call E_χ a *Lyapunov space* or *Lyapunov distribution* for the action. Then the strong stable distribution E_a^- of $a \in \mathbb{R}^k$ is given by $E_a^- = \sum_{\chi(a) < 0} E_\chi$. While individual Lyapunov distributions may not be integrable, the sum $E^\chi = \bigoplus_\lambda E_\lambda$ where λ ranges over all Lyapunov functionals which are positive multiples of a given Lyapunov functional χ is always integrable. In fact, set $H = \{a \in \mathbb{R}^k \mid \chi(a) \leq 0\}$, and call it a *Lyapunov half space*. Then E^χ is the intersection of all stable distributions of elements in H , as is easily seen. We will also denote E^χ by E_H . This is an integrable distribution with integral foliation \mathcal{W}_H^- whose leaves are intersections of strong stable manifolds. Note that \mathcal{W}_H^- is also integrable with the orbits. Denote the resulting foliation by \mathcal{W}_H . Then we get a decomposition

$$TM = \bigoplus E_H \oplus T\mathcal{O}$$

where H runs over all Lyapunov half spaces and $T\mathcal{O}$ is the tangent bundle to the orbits of the action. We call this decomposition the *coarse Lyapunov decomposition* of TM . Note that each E_H is the sum of Lyapunov distributions corresponding to Lyapunov exponents proportional to

each other with positive coefficients of proportionality. Thus for any Lyapunov half space H one can find a uniquely defined Lyapunov characteristic exponent $\chi(H)$ (called the *bottom exponent* for H) and positive numbers $1 = c_1(H) < c_2(H) < \dots < c_{m(H)}(H)$ such that

$$E_H(x) = \bigoplus_{i=1}^{m(H)} E_{c_i(H)\chi(H)}.$$

The situation is particularly simple for algebraic \mathbb{R}^k -actions on quotients of semisimple groups, i.e. for Weyl chamber flows. As was noticed in the previous subsection the only positive coefficient proportionality with a bottom exponent in that case is two. This fact is important for the case of low smoothness mentioned above.

Finally note that the coarse Lyapunov spaces are precisely the maximal nontrivial intersections of stable distributions of elements in \mathbb{R}^k .

2.2.3 Existence and regularity of the orbit equivalence

Step 1: Hölder conjugacy. Now consider an algebraic Anosov action ρ of \mathbb{R}^k on a manifold M . Let \mathcal{O} denote its orbit foliation. Suppose \mathcal{R} is a C^1 -small perturbation of \mathcal{O} . Let a be a normally hyperbolic element of \mathbb{R}^k . Then the 1-parameter group ta defines a vectorfield V on M . Define a perturbation $\tilde{V}(p)$ for any point $p \in M$ by projecting $V(p)$ to $T\mathcal{R}_p$, the tangent space to \mathcal{R} at p . By Hirsch, Pugh and Shub's structural stability theorem, \tilde{V} is normally hyperbolic to a foliation $\tilde{\mathcal{R}}$, invariant under \tilde{V} which is C^1 -close to \mathcal{O} [9, Theorem 7.1]. Since \tilde{V} -invariant foliations C^1 -close to \mathcal{O} are unique by normal hyperbolicity, $\mathcal{R} = \tilde{\mathcal{R}}$, and thus \tilde{V} is normally hyperbolic to \mathcal{R} . Again applying structural stability, there exists a Hölder orbit equivalence ϕ between \mathcal{R} and \mathcal{O} . We may also assume that ϕ is C^∞ along the leaves. In fact, one identifies the image of a leaf under ϕ uniquely by shadowing (this also proves the transversal uniqueness of the orbit equivalence). Then one can map a leaf to its image by intersecting the image of a neighborhood of the zero section in the normal bundle to the leaf with the image leaf. This yields an orbit equivalence smooth along the leaves.

Let us also note here that to prove smoothness of ϕ it suffices to prove transversal smoothness of ϕ . More precisely, pick smooth transversals T_1 and T_2 to \mathcal{O} and \mathcal{R} respectively. Map T_1 to T_2 by sending $t \in T_1$ to the intersection of the leaf of \mathcal{R} through $\phi(t)$ with T_2 . Call ϕ *transversally smooth* if the resulting map from $T_1 \rightarrow T_2$ is C^∞ . Now the above construction shows that transversal smoothness of ϕ implies that ϕ is C^∞ .

Step 2: Invariance of weak stable foliations. Now consider the coarse Lyapunov decomposition of ρ . We will first show that ϕ takes the foliations \mathcal{W}_H into C^1 -close foliations $\tilde{\mathcal{W}}_H$ saturated by \mathcal{R} . Pick normally hyperbolic elements a_1, \dots, a_q such that E_H^- is the intersection of the strong stable distributions $E_{a_s}^-$, $s = 1, \dots, q$. Let V_s denote the vectorfields that generate the flows $\rho(t a_s)$ on M . As above, let \tilde{V}_s be the projection of V_s to the tangent distributions of \mathcal{R} (Note that the \tilde{V}_s do not necessarily commute). Nevertheless, \mathcal{R} is invariant under \tilde{V}_s for all s , and all \tilde{V}_s are normally hyperbolic with respect to \mathcal{R} . As \tilde{V}_s and V_s are C^1 -close, structural stability implies that ϕ maps the weak stable foliations of V_s to those of \tilde{V}_s [9]. As the tangent distributions of the weak stable foliations of \tilde{V}_s are C^0 -close to those of V_s , they still intersect transversely in a foliation $\tilde{\mathcal{W}}_H$ which by definition is saturated under \mathcal{R} . In particular, the $\tilde{\mathcal{W}}_H$ are invariant under the flows of \tilde{V}_s , and in particular under their time 1 maps \tilde{a}_s . Denote their tangent distributions by \tilde{E}_H .

Consider the induced operator a_s^* on the Banach space of sections of the bundle E_H given by $a_s^* v(x) = D\rho(a_s)(v\rho(a_s^{-1}(x)))$. Then the characteristic set of a_s^* just consists of the numbers $\exp \chi(a_s)$ where χ runs over all the Lyapunov exponents with $E_\chi \subset E_H$.

Step 3: Construction of a perturbed action and natural extension. Define a continuous action $\tilde{\rho}$ of \mathbb{R}^k on M by conjugating ρ by ϕ . Next we will construct an extension of this action on the bundle T , defined as follows. Since $\tilde{\mathcal{W}}_H(x)$ is C^1 -close to $\mathcal{W}_H(\phi^{-1}(x))$, we can project $\mathcal{W}_H(\phi^{-1}(x))$ to $\tilde{\mathcal{W}}_H(x)$ by a C^∞ map, close to the identity. In particular, the image of the foliation \mathcal{W}_H^- on $\mathcal{W}_H(\phi^{-1}(x))$ defines a smooth foliation \mathcal{T} with leaves T_y , $y \in \tilde{\mathcal{W}}_H(x)$, such that the leaf T_x is C^0 -close to $\phi(\mathcal{W}_H^-(\phi^{-1}(x)))$. Locally we can identify the bundle of T_x with the normal bundle of \mathcal{R} in $\tilde{\mathcal{W}}_H$. Finally, define the extension of $\tilde{\rho}$ by holonomy as follows. If $y \in T_x$ and $a \in \mathbb{R}^k$, let $\mathcal{A}(y)$ be the unique intersection point of the local leaf of $\mathcal{R}(\tilde{\rho}(a)(y))$ with $T_{\tilde{\rho}(a)(x)}$.

Let \mathcal{A}_s denote the extensions of $\tilde{\rho}(a_s)$ on T . These extensions commute with each other. Define the operators \mathcal{A}_s^* on the Banach space of continuous sections of T , endowed with the uniform norm, as usual. Since the extensions \mathcal{A}_s are small perturbations of the $\rho(a_s)$, the spectra of the \mathcal{A}_s^* are close to those of a_s^* . Thus the characteristic set of \mathcal{A}_s^* is contained in small intervals $\Delta_i = [\lambda_i, \mu_i]$ about the characteristic set of a_s^* where we order the intervals in increasing order such that $\lambda_{i+1} > \mu_i$. The size of the intervals depends on the size of the perturbation of ρ . Thus we may assume that the operator \mathcal{A}_s^* has narrow band spectrum (cf. 2.2.1).

Step 4: Smoothness along coarse Lyapunov directions - the main step. We will now show that $\phi : \mathcal{W}_H^- \rightarrow \tilde{\mathcal{W}}_H$ is smooth. Let $\psi_x : \mathcal{W}_H^-(x) \rightarrow T_{\phi(x)}$ denote the composition of ϕ and projection to $T_{\phi(x)}$ along the leaves of \mathcal{R} .

As we will see, these ψ_x intertwine the smooth transitive actions of a Lie group G on both $\mathcal{W}_H^-(x)$ and T_x . Then conjugation by ψ_x defines a continuous and hence C^∞ homomorphism $\eta_x : G \rightarrow G$. Since $\psi_x(gx) = \eta_x(g)\psi_x(x)$, we see immediately that ψ_x is C^∞ .

Let us first define G and its action on $\mathcal{W}_H^-(x)$ for a.e. $x \in M$. Let χ be a Lyapunov exponent such that H is the halfspace on which χ is nonpositive. Since $\mathcal{W}_H^-(x)$ is a maximal nontrivial intersection of stable manifolds, by the hypothesis of the theorem there is a one-parameter subgroup tb such that $\chi(b) = 0$ and whose ergodic components are saturated by leaves of \mathcal{W}_H^- . Let M be the quotient of a compact homogeneous H/Λ by a compact group K commuting with an affine action ρ^* of \mathbb{R}^k such that ρ is the quotient action. Then we can endow H with a right invariant metric such that ρ^* dilates this metric according to the linear part σ^* of ρ^* and K leaves the metric invariant. Let \mathbf{g} be the induced Riemannian metric on M . Then the maps $\rho(tb) : \mathcal{W}_H^-(x) \rightarrow \mathcal{W}_H^-(\rho(tb(x)))$ are isometries with respect to the Riemannian metrics induced by \mathbf{g} on the $\mathcal{W}_H^-(x)$.

Since the ergodic components of $\rho(tb)$ are saturated by leaves of \mathcal{W}_H^- , for a.e. point x , the closure of the orbit $\rho(tb)(x)$ contains $\mathcal{W}_H^-(x)$. Fix such a point x . In particular, given any $y \in \mathcal{W}_H^-(x)$, there is a sequence of t_n such that $\lim_{n \rightarrow \infty} \rho(t_nb)(x) = y$. We may assume moreover, that the sequence of isometries $\rho(t_nb) : \mathcal{W}_H^-(x) \rightarrow \mathcal{W}_H^-(\rho(t_nb(x)))$ converges to an isometry $g : \mathcal{W}_H^-(x) \rightarrow \mathcal{W}_H^-(x)$ which takes x to y . In fact, it is easy to see that such a limit is a local isometry and also a covering map. Since $\mathcal{W}_H^-(x)$ is simply connected, it follows that the limit is a global isometry (cf. [8, Proposition 2.9] for a detailed proof). Let $G = G_x$ be the closure of the group generated by all such limits g in the isometry group of $\mathcal{W}_H^-(x)$. Then G acts transitively on $\mathcal{W}_H^-(x)$.

Since $\psi_x : \mathcal{W}_H^-(x) \rightarrow T_x$ is a homeomorphism, G acts on T_x by conjugating by ψ_x . We will show that for all $g \in G$, g acts smoothly on T_x . Let us first show this for a limit g of isometries $g_n = \rho(t_nb) : \mathcal{W}_H^-(x) \rightarrow \mathcal{W}_H^-(\rho(t_nb(x)))$. Then $h_n := \psi_{g_n x} g_n \psi_x^{-1}$ is a compact family of maps in the C^0 -topology, converging to a map $h : T_{\phi(x)} \rightarrow T_{\phi(x)}$. Note that h is g acting on $T_{\phi(x)}$. On the

other hand, the h_n are C^∞ -maps, as in fact they are holonomy maps for \mathcal{R} .

Now comes the central point in the argument. Since natural extension \mathcal{A} of the conjugated action $\tilde{\rho}$ contains a contraction \mathcal{A}_s satisfying the narrow band condition, by Corollary 10 we can find a family of C^∞ coordinate changes in the fibers T_x , depending continuously on x , such that the \mathcal{A}_s , and therefore the h_n , which commutes with \mathcal{A}_s , written in these coordinates act as elements of a certain finite dimensional Lie group, namely the group $SR_{\lambda,\mu}$. As the h_n are compact in C^0 , they are also a compact family in $SR_{\lambda,\mu}$. Therefore, the h_n converge in the C^∞ -topology, and h is smooth, and belongs to $SR_{\lambda,\mu}$. Moreover, the group generated by such h as well as its closure G belongs to $SR_{\lambda,\mu}$. Thus every element in G acts smoothly on T_x .

Now suppose that y is a limit of points x_n such that $\mathcal{W}_H^-(x_n)$ is contained the orbit closure of x_n under $\rho(tb)$. Then the special Lie group of isometries G_{x_n} constructed above is transitive on $\mathcal{W}_H^-(x_n)$, and acts smoothly on T_{x_n} by diffeomorphisms in $SR_{\lambda,\mu}$. Furthermore, ψ_{x_n} conjugates these two actions. Let G_y be the group generated by all C^0 -limits of $g_n \in G_{x_n}$. Then G_y is a transitive group of isometries of $\mathcal{W}_H^-(y)$. Let $g = \lim g_n$ be such a limit. Then $\psi_{x_n} \circ g_n \circ \psi_{x_n}^{-1}$ converges to $\psi_y \circ g \circ \psi_y^{-1}$ in the C^0 -topology. Since the $\psi_{x_n} \circ g_n \circ \psi_{x_n}^{-1}$ again belong to $G_{\lambda,\mu}$, this convergence is again C^∞ . Finally, let us note that any $y \in M$ is such a limit since the union of the ergodic components of $\rho(tb)$ have full measure.

Thus we have shown that for all $y \in M$, there is a Lie group G_y of isometries of $\mathcal{W}_H^-(y)$ which acts continuously on the transversals T_y by diffeomorphisms and such that ψ_y conjugates these actions. By a theorem of Montgomery, the G_y action on the T_y is C^∞ [30, Section 5.1, Corollary]. As we mentioned at the beginning of this step of the proof, this is sufficient to conclude that ψ_y is a C^∞ diffeomorphism since ψ_y intertwines two smooth transitive actions of a Lie group.

In the case of C^6 (of C^4 in the absence of double roots) perturbations of Weyl chamber flows we use instead of Corollary 10 the C^3 (corr. C^2) normal form and the rigidity of the C^3 (corr. C^2) centralizer established by Guysinsky (see the end of Section 2.2.1). The conclusion is that the conjugation intertwines two transitive Lie groups of diffeomorphisms of corresponding finite regularity. The finite regularity version of the Montgomery theorem implies then that the conjugacy is at least C^1 along \mathcal{W}_H^- .

Step 5: Global smoothness.

We will now show that ϕ is C^∞ . The manifold M is foliated by the smooth foliations \mathcal{W}_H^- and \mathcal{O} whose tangent bundles are transverse and span the TM . We showed above that ϕ is C^∞ along all the foliations \mathcal{W}_H^- . As ϕ is already C^∞ along \mathcal{O} by construction of ϕ , then ϕ is smooth along a full set of directions. Now it follows from standard elliptic operator theory that ϕ is a C^∞ -function (as the elliptic operator $\frac{\partial^{2s}}{\partial x_1^{2s}} + \dots \frac{\partial^{2s}}{\partial x_n^{2s}}$ sends ϕ to a continuous function where x_1, \dots, x_n are coordinates subordinate to the foliations above.) Global C^1 regularity in the finite differentiability case directly follows from the C^1 regularity along each foliation.

This finishes the proof of Theorem 1. ◇

2.3 Proof of Corollaries to Theorem 1

2.3.1 Proof of Corollary 2

It suffices to check the conditions of Theorem 1. Since the maximal nontrivial intersections of stable manifolds are exactly coarse Lyapunov spaces and all one-parameter subgroups are ergodic, we can pick tb to be any one-parameter subgroup in the boundary of the Lyapunov halfspace.

2.3.2 Proof of Corollary 3

The argument for the Weyl chamber flows has been given before the statement of the corollary in Section 2.1.

We will use the notation established in the description of twisted Weyl chamber flows before Corollary 3. It suffices to prove that all one-parameter subgroups of A are ergodic. If G is simple, as above, Moore's theorem yields the ergodicity of all one-parameter subgroups so that we can apply Corollary 2 above. For the general case, we will need to use the results of Brezin and Moore. Let us briefly recall some notions from [4]. Call a simply connected Lie group *Euclidean* if it is the universal cover of an extension of a vector group by a compact abelian group. A solvmanifold is called *Euclidean* if it is the quotient of a Euclidean group by a closed subgroup. Further, given a connected Lie group H and a closed subgroup D , we call the pair (H, D) *admissible* if D normalizes a closed solvable subgroup containing the radical of H . Note that the pair $(G \ltimes N, \Gamma \ltimes \Lambda)$ is always admissible. Brezin and Moore show that every admissible pair (H, D) has a maximal Euclidean quotient H/L where $D \subset L$. Let $H^{(n)} = [H^{(n-1)}, H^{(n-1)}]$ denote the n -th element of the derived series where $H^{(0)} = H$. Then $H^{(n)}$ will eventually be constant. Let $H^{(\infty)}$ denote this subgroup. Brezin and Moore show that $H^{(\infty)} \subset L$. For our special case of $H = G \ltimes N$, where the twisting homomorphism contains an Anosov element, the derived series is constant right away, as follows from a consideration of eigenvalues. Hence the maximal Euclidean quotient is a point.

Brezin and Moore show that a one parameter subgroup of H acts ergodically on H/D for an admissible pair (H, D) if and only if the quotient flows on the maximal Euclidean quotient and the maximal semisimple quotient are ergodic. The latter quotient is obtained by factoring H by the closure of DR where R is the radical of H . In our case, the latter is isomorphic to G/Γ as $\Gamma \ltimes N$ is closed in $G \ltimes N$. Since Γ is irreducible, all one-parameter subgroups of A are ergodic on G/Γ . Since the maximal Euclidean quotient is a point, we get ergodicity of one-parameter subgroups of A .

Remark. A. Starkov pointed out to us an alternative argument, avoiding the discussion of admissibility, which uses Theorem 8.24 from [40] and certain results proved independently by himself and by D. Witte.

2.3.3 Proof of Corollary 4

Suppose \mathbb{Z}^k acts on a compact nilmanifold M . Then the suspension $(M \times \mathbb{R}^k)/\mathbb{Z}^k$ factors over $T^k := \mathbb{R}^k/\mathbb{Z}^k$ with fiber M . Since the strong stable manifolds in the suspension of any $b \in \mathbb{R}^k$ are contained in the fibers, the condition on intersections of unstable manifolds in Theorem 1 is satisfied if we know that the ergodic components of any one-parameter subgroup ta of the suspension consist of whole fibers. It clearly suffices to check this on some finite cover of M . Thus, we will assume that $M = N/\Lambda$ is a nilmanifold, and N is a simply connected nilpotent group.

Note that the linearization of the \mathbb{Z}^k -action extends (in general, not uniquely) to \mathbb{R}^k . Let $S = \mathbb{R}^k \ltimes N$ be the solvable group defined by this extension. Then $\Sigma := \mathbb{Z}^k \ltimes \Lambda$ is a lattice in S , and the natural fibration of S/Σ over T^k is just the fibration defined by the suspension. Furthermore, the action of a one-parameter subgroup in the suspension is just the homogeneous action of a one-parameter subgroup of \mathbb{R}^k on S/Σ via the embedding of \mathbb{R}^k into S .

As in the proof of the previous corollary we will now apply the results of J. Brezin and C. C. Moore on homogeneous flows [4]. For a general compact solvmanifold, Brezin and Moore show that there always exists a maximal quotient which is Euclidean. Moreover, this quotient is unique and is called the *maximal Euclidean quotient*. They further show that a homogeneous flow on a solvmanifold is ergodic if the quotient flow on the maximal Euclidean quotient is ergodic [4,

Theorem 6.1].

For the case of our suspension flow, our claim about ergodic components will follow quickly once we determine the maximal Euclidean quotient of suspensions.

Lemma 11 *The maximal Euclidean quotient of the suspension S/Σ of an algebraic \mathbb{Z}^k -action for which every nontrivial element is weakly mixing is T^k .*

Assume the lemma. Let T^r be an ergodic component of the quotient action on T^k of some one-parameter group $ta \in \mathbb{R}^k$. Then $T^r = \mathbb{R}^r/\mathbb{Z}^r$ for $\mathbb{Z}^r \subset \mathbb{Z}^k$. Then the preimage of T^r in S/Σ is a solvmanifold S^r/Σ^r , and in fact is the suspension of the \mathbb{Z}^r action on M . Applying the lemma to S^r/Σ^r , the flow on S^r/Σ^r is ergodic by Brezin's and Moore's theorem. Hence the ergodic components of the one-parameter group on S/Σ are all preimages of subtori of T^k , and in particular contain whole fibers.

Proof of the Lemma: Let S/D be the maximal Euclidean quotient of S/Σ . Since T^k is a Euclidean quotient, $D \subset \mathbb{Z}^k \ltimes N$. We are done if the connected component of the identity D^0 of D is N . Suppose not. Then D^0 does not project onto $N/[N, N]$ since a subspace of the Lie algebra of N complementary to the commutator algebra generates the whole Lie algebra. Also note that the linear part of the \mathbb{Z}^k -action on N/D^0 has eigenvalues of modulus 1 since they correspond to eigenvalues of the flow on the maximal Euclidean quotient.

Since $D \cap N \supset \Sigma \cap N = \Lambda$, $D \cap N$ is cocompact, and hence Zariski dense in N (cf. e.g. [40, Theorem 2.3]). Thus $D^0 \subset D \cap N$ is normal in N . Since $\mathbb{Z}^k \subset D$, \mathbb{Z}^k normalizes D^0 . Since the normalizer of D^0 is algebraic it contains a proper R^k in which our \mathbb{Z}^k is a lattice. We conclude that D^0 is normal in S . Hence D/D^0 is a lattice in S/D^0 , or equivalently, $\mathbb{Z}^k \ltimes (D \cap N)/D^0$ is a lattice in $\mathbb{R}^k \ltimes N/D^0$. This implies that \mathbb{Z}^k leaves the lattice $(D \cap N)/D^0$ in N/D^0 invariant. It follows that \mathbb{Z}^k leaves the lattice $(D \cap N)/D^0 [N, N]$ in $N/D^0 [N, N]$ invariant. Since \mathbb{Z}^k acts on $N/D^0 [N, N]$ with linearization with eigenvalues of modulus 1, these eigenvalues have to be roots of unity. Thus the linearization of \mathbb{Z}^k on $N/[N, N]$ also has roots of unity as eigenvalues unless $D^0 = N$. By Parry's characterization of weakly mixing automorphisms of nilmanifolds, no nontrivial element of \mathbb{Z}^k can have eigenvalues of modulus 1 on $N/[N, N]$, which is a contradiction [31].

2.3.4 Proof of Corollary 5

First we establish that any C^1 -small perturbation of a standard Anosov action is C^∞ -orbit equivalent to the original action. In the case of the suspension of an action by automorphisms of an infranilmanifold this follows from Corollary 4 and in the cases of Weyl chamber flows and twisted Weyl chamber flows from Corollary 3. The remaining extension cases are handled similarly and we omit the necessary but somewhat tedious algebraic arguments. Hence the perturbed action is smoothly conjugate to a C^∞ -time change of the standard Anosov action. However, we showed in [21] that such time changes are C^∞ -conjugate to the original action up to an automorphism.

Furthermore, as the orbit equivalence in question can be chosen C^1 -close to the identity, the resulting time change is close to the identity. One can further choose the conjugacy between the time change and the original action close to the identity. In fact, we solve a cocycle equation in [21] by defining the coboundary P as a sum of the cocycle over the forward orbit. It follows easily that the derivative is small along stable manifolds, and similarly along unstable manifolds. Hence the coboundary is close to 0 in the C^1 -topology if it is close to 0 at some point. As we can always pick the coboundary to have 0 average with respect to the invariant volume, P is close to 0 somewhere.

That the automorphism ρ is close to the identity and unique follows as it is an average of the cocycle determined by the time change over the given volume.

Finally, suppose that ϕ_1 and ϕ_2 are two conjugacies close to the identity. Then $\phi = \phi_2^{-1} \phi_1$ commutes with the \mathbb{R}^k -action and is close to the identity. Since the identity is an orbit equivalence of the \mathbb{R}^k -action with itself, ϕ has to take orbits to themselves by the uniqueness part of Theorem 1. Now fix a point p with a dense \mathbb{R}^k -orbit. This is possible as the action is ergodic. Then find the translation $a \in \mathbb{R}^k$ such that $a p = \phi(x)$. Then a and ϕ coincide on the \mathbb{R}^k -orbit of p , and hence everywhere as that orbit is dense. \diamond

2.4 The reductive case

Let us make a few remarks about the changes needed to prove Theorem 6 and Corollary 7. Suppose we have an Anosov action of a reductive group $H = LC$ on G/Λ where G is a semisimple Lie group, C is a subgroup of a split Cartan, and $\Lambda \subset G$ is a lattice. First note that the orbit foliation \mathcal{O} of H is the neutral foliation of a suitable element $a \in C$. Let \mathcal{R} be a small perturbation of \mathcal{O} . Perturbing vectorfields from C as before, we can recover \mathcal{R} as a normally hyperbolic invariant foliation for some perturbed vector field. By structural stability, we get an orbit equivalence. Now we proceed as in the abelian case, and define a continuous action of C which is subordinate to \mathcal{R} . The difference to the abelian case is that this action is not transitive on \mathcal{R} anymore. Still, it is ergodic and hence topologically transitive. Hence we get transitive groups on the leaves of the coarse Lyapunov foliations, as before. The rest of the arguments apply verbatim.

Corollary 7 follows the same way, as in fact the group C still acts on $\Gamma \backslash G/M_P$.

2.5 Automorphisms of Weyl chamber flows

For an action α of a group G on a manifold M let the *automorphism group* $Aut(\alpha)$ be the centralizer of $\alpha(G)$ in the homeomorphism group of M .

In this subsection we prove a result showing that a Weyl chamber flow is a maximal abelian action on its ambient manifold.

Theorem 12 *If α is a Weyl chamber flow on $K \backslash H/\Lambda$ where H is a semisimple Lie group of rank $k \geq 2$ without compact factors, then the automorphism group of α consists of C^∞ -diffeomorphisms and is a finite extension of \mathbb{R}^k , i.e. the \mathbb{R}^k*

action itself is a subgroup of finite index in the centralizer.

This result has more to do with semisimplicity of H than with the rank assumption which is central for all the rest of our results. For example, the assertion is also well-known for the Anosov actions of the split Cartan subgroups of rank one semisimple Lie groups G of the non-compact type on a compact quotient $M \backslash G/\Gamma$ where Γ is a cocompact lattice and M is the compact part of the centralizer of a split Cartan subgroup.

On the other hand, if α is an Anosov action on a nilmanifold, then all its automorphisms are diffeomorphisms as is well known but the centralizer naturally may be much bigger than the action. The situation is similar for twisted Weyl chamber flows which are toral fiber bundles over a Weyl chamber flow. We will not discuss this case here however.

Proof: Let α be a Weyl chamber flow. Let $N = K \backslash H/\Lambda$ be the space α acts on. Suppose that $\phi \in Aut(\alpha)$. Let $\Gamma = \pi_1 N$, the fundamental group of N . Then ϕ lifts to a Γ -equivariant map on the universal cover \tilde{N} of N . Note that \tilde{N} is a homogeneous space of the universal cover \tilde{H} of H

H and let $\rho : \tilde{H} \rightarrow \tilde{N}$ be the factor map. Lift \mathbb{R}^k to \tilde{H} . The space of weak stable manifolds of $a \in \mathbb{R}^k$ on \tilde{N} is isomorphic to the space of weak stable manifolds of $a \in \mathbb{R}^k$ on \tilde{H} . The latter can be described as the set of parabolics H/P^a where the Levi component of P^a is the centralizer of $\log a$ in \tilde{H} . Thus we can map $x \in \tilde{N}$ to the parabolic $\sigma_a(x)$ which is the stabilizer of the weak stable manifold $\rho^{-1}(W_a^{ws}(x))$ of a at any point in $\rho^{-1}(x)$.

Note that for all $g \in \tilde{G}$, $\sigma_a(xg) = g\sigma_a(x)g^{-1}$. Hence σ_a is surjective onto H/P^a . If $\sigma_a(x) = \sigma_a(y)$ then $y \in W_a^{ws}(x)$ as follows from the equivariance above and the self-normalizing property of P^a . Therefore and because ϕ takes weak stable manifolds of a to weak stable manifolds of a we can define a map $\bar{\phi}_a : H/P^a \rightarrow H/P^a$ by

$$\bar{\phi}_a \circ \sigma_a = \sigma_a \circ \tilde{\phi}.$$

Let a belong to an open face F of a Weyl chamber. If b belongs to the boundary of F then

$$\sigma_a(x) \subset \sigma_b(x).$$

Thus if $P_1 \subset P_2$ are parabolics in H/P_a and H/P_b respectively, then

$$\phi_a(x) \subset \phi_b(x).$$

Therefore the ϕ_a define an automorphism of the Tits building Δ attached to \tilde{H} where a ranges over a closed Weyl chamber.

We will show now that σ_a and ϕ_a are independent of a for all regular elements $a \in \mathbb{R}^k$. First note that they do not change as a varies over an open face of a Weyl chamber since the weak stable manifolds stay the same. If a and a' belong to adjacent open Weyl chambers \mathcal{C} and \mathcal{C}' then $\phi_a = \phi_{a'}$. Indeed, ϕ_a and $\phi_{a'}$ coincide on the faces of the type of the intersection of $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}'$ by equivariance. Thus they coincide on all of Δ by a convexity argument. Since any two Weyl chambers are connected by a chain of adjacent Weyl chambers, it follows that all ϕ_a coincide.

Thus $\tilde{\phi}$ defines a unique automorphism $\bar{\phi}$ of Δ which is clearly continuous for the topology on Δ inherited from \tilde{H} . Note that $\tilde{\phi}$ is completely determined by $\bar{\phi}$ transversally to the \mathbb{R}^k -orbits by projecting $x \in \tilde{N}$ simultaneously to $\sigma_a(x)$ and $\sigma_{-a}(x)$ for a regular element a . Since a topological automorphism of Δ is C^∞ (for the smooth structure on Δ induced by the \tilde{G} -action) [5] and the maps σ_a and σ_{-a} are C^∞ we see that $\tilde{\phi}$ is C^∞ transversally to the \mathbb{R}^k -orbits. As $\tilde{\phi}$ is C^∞ along the orbits, $\tilde{\phi}$ is C^∞ .

Consider the group $\tilde{Aut}(\alpha)$ of lifts of automorphisms of α to \tilde{N} and the map $\tilde{Aut}(\alpha) \rightarrow Aut(\Delta)$ where $Aut(\Delta)$ is the group of topological automorphisms of Δ . Then \mathbb{R}^k embeds canonically into $\tilde{Aut}(\alpha)$, and is the kernel of the map into $Aut(\Delta)$. Indeed, if $\bar{\phi} = id$ then $\tilde{\phi}$ maps every \mathbb{R}^k -orbit in \tilde{N} into itself as $\tilde{\phi}$ is transversally determined by $\bar{\phi}$. Then we can write uniquely $\tilde{\phi}(x) = a(x)x$ for $x \in \tilde{N}$, and thus $\phi(x) = a(x)x$ at least for all $x \in N$ without isotropy in \mathbb{R}^k . Since ϕ is an automorphism, $a(x)$ is constant along orbits, and hence constant by ergodicity of the \mathbb{R}^k -action. By [5], the topological automorphism group of a topological building is a finite extension of $\text{Ad } G$. Hence the image of $\tilde{Aut}(\alpha)$ in $Aut(\Delta)$ contains the intersection I with $\text{Ad } H$ as a subgroup of finite index. If $\bar{\phi} \in I$ then $\bar{\phi}$ belongs to the normalizer of $\text{Ad } \Lambda$ in $\text{Ad } H$ since $\bar{\phi}$ is Λ equivariant. Further, if $\bar{\phi} \in \text{Ad } \Lambda$ then $\phi = id$. As the group of outer automorphisms of Λ is finite (as follows e.g. from Mostow's rigidity theorem), the theorem is proved.

3 Algebraic Anosov actions of lattices

We will use our results on smooth local rigidity of affine Anosov actions of abelian groups to investigate algebraic Anosov actions of lattices in semisimple Lie groups of higher rank. In particular, we obtain local rigidity for general affine Anosov actions of such lattices in Theorem 15 below.

We will first prove a general result, Proposition 13, asserting the local rigidity of certain affine lattice actions modulo the local rigidity of suitable large Abelian subgroups. Theorem 15 itself follows albeit not so quickly from this and our results in Section 2. The main two steps involve verifying the assumptions of Proposition 13 and passing from rigidity on a subgroup of finite index to the rigidity of the whole action. Proposition 13 may prove useful in establishing local rigidity results beyond the Anosov case.

Let G be a linear semisimple Lie group without compact factors all of whose simple factors have real rank at least 2. Let Γ be an irreducible lattice in G . Call a subgroup $\Delta \subset \Gamma$ a *Cartan subgroup* of Γ if there is a Cartan subgroup D in G with maximal \mathbb{R} -split component such that Γ intersects D in a lattice in D . In particular, Δ is an abelian group whose rank is the real rank of G . By a theorem of G. Prasad and M. Raghunathan, any lattice Γ contains Cartan subgroups [32, Theorem 2.8].

Let M be a manifold of dimension N with a C^∞ -action ρ of a group Γ preserving a probability measure μ . We call a measurable framing τ of a manifold M a *superrigidity framing* for ρ if there is a homomorphism $\sigma : \Gamma \rightarrow GL(N, \mathbb{R})$, a compact subgroup $C \subset GL(N, \mathbb{R})$ commuting with the image of σ and a measurable cocycle $\kappa : \Gamma \times M \rightarrow C$ such that for all $\gamma \in \Gamma$ and μ -a.e. $x \in M$

$$D\rho(\gamma)(\tau(x)) = \tau(\gamma x) \sigma(\gamma) \kappa(\gamma, x).$$

Then π is called a *superrigidity representation*, and κ a *superrigidity cocycle*. Furthermore, we call a framing τ on a homogeneous space L/Λ *translation invariant* if τ lifts to a framing of L invariant under right translation by L .

Proposition 13 *Let G be a linear semisimple Lie group without compact factors all of whose simple factors have real rank at least 2. Let Γ be an irreducible lattice in G , Δ a Cartan subgroup of Γ , and ρ an ergodic affine algebraic action of Γ on a manifold M . Assume that the restriction of ρ to Δ is locally C^∞ -rigid, and that we can choose the conjugacy C^1 -close to the identity. Assume further that for some $a \in \Delta$, $\rho(a)$ as well as all lifts of finite powers to finite connected covers are weakly mixing. Then any C^∞ -action $\tilde{\rho}$ of Γ , sufficiently close to ρ in the C^1 -topology, is C^∞ -conjugate to an action with a translation invariant superrigidity framing for a subgroup Γ_0 of Γ of finite index.*

Let us emphasize that the affine lattice action need not be Anosov in this proposition.

Proof: The basic idea of the proof is to compare a superrigidity framing for the perturbed action with a translation invariant framing using the dynamics of the Cartan subgroup Δ and see how the framings get transformed under Δ . In effect, we prove a certain uniqueness of superrigidity framings for suitable affine actions of Δ .

Step 1: Measurable superrigidity We first need to review some aspects of R. Zimmer's superrigidity theorem for cocycles [50]. Suppose $\tilde{\rho}$ is a Γ -action on a manifold M preserving a finite measure on M . As G is linear, G is a quotient of the maximal algebraic factor \bar{G} of the universal cover of G . Lifting Γ to a lattice in \bar{G} , we may assume without loss of generality that G is the maximal algebraic factor of its universal cover. Let $P \rightarrow M$ be a principal bundle over M with structure group L on which Γ acts smoothly by bundle automorphisms covering the action on the base. We will denote this extension $D\tilde{\rho}$ since in our application it will be generated by the differentials of a smooth action in the base. The *algebraic hull* of P and $\tilde{\rho}$ is the smallest algebraic subgroup H of L (up to conjugacy), for which there is a measurable section τ of P which transforms under $\tilde{\rho}$ by some elements $\alpha(\gamma, x)$ of H :

$$D\tilde{\rho}(\gamma)(\tau(x)) = \tau(\tilde{\rho}(\gamma)(x)) \alpha(\gamma, x).$$

Then α is a cocycle over $\tilde{\rho}$ with values in H . Furthermore, H is always reductive with compact center as was shown for uniform lattices by R. Zimmer in [51] and for non-uniform lattices by J. Lewis in [27]. In order to be able to apply Zimmer's superrigidity theorem for cocycles [50], the algebraic hull needs to be (Zariski) connected. This can be achieved by passing to a suitable finite cover M' of M as follows. Let H^0 be the Zariski connected component of H . Let P be the reduction of the frame bundle of M with structure group H . Set $M' = \frac{P \times H^0 \backslash H}{H}$. Then $M' \rightarrow M$ is a finite cover, on which Γ acts as a quotient of the bundle automorphisms on P by H . Note that this lift of the Γ -action preserves a finite volume. Moreover, $P \rightarrow M'$ is a principal H^0 -bundle on which Γ acts via bundle automorphisms. Measurably, M' is $M \times H/H^0$. The action is a skew product action. Let β be the projection of α to H/H^0 , and define the skew product action of Γ on M' by

$$\gamma(x, [h]) = (\gamma x, \beta(\gamma, x)[h]).$$

Note also that the skew product leaves the product of the invariant volume on T^N and counting measure invariant. Now the following theorem is the desired corollary of measurable superrigidity (cf. e.g. [39, Theorem 3.1]).

Theorem 14 *Passing to the finite cover M' of M described above, there is a measurable section τ^* of P which transforms under $\tilde{\rho}$ by a cocycle of the form $\pi(\gamma) \kappa(\gamma, x)$ where $\pi : G \rightarrow H^0$ is a homomorphism, and κ is a cocycle taking values in a compact normal subgroup of H^0 centralizing $\pi(\Gamma)$.*

Let us note here that M' is only constructed in the category of measure spaces since the reduction P of the frame bundle with structure group H a priori is only measurable. This will cause some complications in the next step of the proof.

Step 2: Regularity of the superrigidity framing We assume the notations from the statement of Proposition 13. As ρ is affine, M is the quotient of a connected Lie group L , $M = L/\Lambda$. We will show that if $\tilde{\rho}$ is sufficiently close to ρ in the C^1 -topology, then $\tilde{\rho}$ is C^∞ -conjugate to ρ . How close $\tilde{\rho}$ needs to be to ρ will be specified during the proof. Since the restriction of ρ to Δ is locally C^1 -rigid, and since we may choose the conjugacy C^1 -close to the identity, we may assume that ρ and $\tilde{\rho}$ coincide on the Cartan subgroup Δ .

Since Γ has Kazhdan's property and ρ preserves Haar measure, a perturbation $\tilde{\rho}$ sufficiently close to ρ in the C^1 -topology preserves an absolutely continuous probability measure (cf. e.g. [20, Lemma 2.6] or [43]).

Now let us apply Theorem 14 to $\tilde{\rho}$ and its extension by derivatives to the frame bundle of M . In particular, we let M' denote the finite cover of M for the $\tilde{\rho}$ action and τ^* be a superrigidity framing of M' as in Theorem 14. Denote the action of Γ on M' extending $\tilde{\rho}$ by $\hat{\rho}$.

Let τ be a translation invariant framing of M . We will now show that we can pick a superrigidity framing for a subgroup of finite index of Γ on M' which is just a constant translate of τ modulo a compact subgroup.

Since M' a priori is only a measurable gadget and our arguments involve some topology, we will need to argue on the base manifold M . Thus we will consider the given superrigidity framing τ^* as a finite collection of frames $\tau^*(x)$ over any point x in M of cardinality $c = \#(H/H^0)$ corresponding to the c points in M' which cover x . Let \mathcal{G} denote the space of unordered c -tuples of points in $GL(N, \mathbb{R})$. Then we can define a measurable function $\beta : M \rightarrow \mathcal{G}$ such that $\tau(x) \beta(x) = \tau^*(x)$. We can think of β as a multivalued function with values in $GL(N, \mathbb{R})$.

Since the restriction of $\hat{\rho}$ to Δ is just the original action ρ on M , the framing τ is transformed under Δ by σ . To simplify notations, we will denote the action of $\hat{\rho}(b)$ by bx for $b \in \Delta$. By assumption, there is an element $a \in \Delta$ which is weakly mixing on M . We will also use the symbol d for the bundle extension.

Since the translation invariant framing τ is transformed under Δ by the automorphism σ , we get for $a \in \Delta$ and $x \in M$:

$$da(\tau^*(x)) = da(\tau(x)\beta(x)) = \tau(ax)\sigma(a)\beta(x)$$

where $g \in GL(N, \mathbb{R})$ acts on \mathcal{G} in two ways by either multiplying each element of the c -tuple by g on the left or the right. On the other hand, by Theorem 14 we have for a.e. x ,

$$da(\tau^*(x)) = \tau^*(ax)\pi(a)\kappa(a, x) = \tau(ax)\beta(ax)\pi(a)\kappa(a, x).$$

Hence we get for a.e. x and all $a \in \Delta$,

$$\sigma(a)\beta(x) = \beta(ax)\pi(a)\kappa(a, x).$$

By Lusin's theorem, for every $\varepsilon > 0$, there exists a closed subset $M_\varepsilon \subset M$ such that the measure of M_ε is at least $1 - \varepsilon$ and β is uniformly continuous on M_ε . Pick $a \neq 1 \in \Delta$. As the transformation induced by a on M_ε preserves the induced measure, a.e. point $x \in M_\varepsilon$ is recurrent. Consider such a point x . Then there is a sequence of integers $n_k \rightarrow \infty$ such that $a^{n_k}x \in M_\varepsilon$ and $a^{n_k}x \rightarrow x$. Let b_0 be an element of $\beta(x)$. Then there are elements $b_{n_k} \in \beta(a^{n_k}(x))$ such that $\sigma(a)^{n_k}b_0 = b_{n_k}\pi(a)^{n_k}\kappa(a^{n_k}, x)$. Since the values of κ and the image of π commute, it follows that

$$b_0^{-1}\sigma(a^{n_k})b_0 = b_0^{-1}b_{n_k}\kappa(a^{n_k}, x)\pi(a^{n_k}).$$

Since $\beta(a^{n_k}x)$ converges to $\beta(x)$, it follows that $b_0^{-1}b_{n_k}$ form a bounded sequence of matrices. Moreover $\kappa(a^{n_k}, x)$ lie in a compact group. Therefore, the eigenvalues of $\sigma(a)$ and $\pi(a)$ have the same absolute values. In other words, they have the same Lyapunov exponents. Moreover the dimensions of the sum of eigenspaces with fixed absolute value (the Lyapunov spaces) coincide. Since a is semisimple, both $\sigma(a)$ and $\pi(a)$ are semisimple. Hence we can write them as commuting products $\sigma(a) = \sigma_c(a)\sigma_{nc}(a)$ and $\pi(a) = \pi_c(a)\pi_{nc}(a)$ where $\sigma_{nc}(a)$ ($\pi_{nc}(a)$) has only positive eigenvalues and $\sigma_c(a)$ ($\pi_c(a)$) has only eigenvalues of modulus 1. It follows from the discussion above that $\sigma_{nc}(a)$ and $\pi_{nc}(a)$ are conjugate in $GL(N, \mathbb{R})$. Let us also note that these decompositions are unique. Hence, any element commuting with $\sigma(a)$ or $\pi(a)$ also commutes with their components.

Pick an element $e \in GL(N, \mathbb{R})$ such that $e\sigma_{nc}(a)e^{-1} = \pi_{nc}(a)$. Set $A = e^{-1}\pi_c(a)e$ and $\bar{\kappa}(a, x) \stackrel{\text{def}}{=} e^{-1}\kappa(a, x)e$. Note that $\sigma_{nc}(a) = e^{-1}\pi_{nc}(a)e$ commutes with both A and the images of $\bar{\kappa}$. Also set $\bar{\tau} = \tau^*e$. Then $\bar{\tau}$ is a measurable superrigidity framing on M' for $\hat{\rho}$ for all of Γ for the representation $e^{-1}\pi e$ and the cocycle $\bar{\kappa}$. Note that $\bar{\tau}$ gives rise to a measurable reduction of the frame bundle which is just a constant translate of P . Hence we may assume that P is the measurable reduction determined by $\bar{\tau}$. Again, we will interpret $\bar{\tau}(x)$ as an unordered c -tuple of frames for x in the base manifold M . Let $\psi : M \rightarrow \mathcal{G}$ be the measurable function such that $\bar{\tau}(x) = \tau(x)\psi(x)$ for $x \in M$. Then we get for all $n \in \mathbb{Z}$

$$\begin{aligned} da^n(\bar{\tau}(x)) &= da^n(\tau^*(x)e) = \tau^*(a^n x)\pi(a^n)\kappa(a^n, x)e = \bar{\tau}(a^n x)e^{-1}\pi(a^n)\kappa(a^n, x)e \\ &= \bar{\tau}(a^n x)e^{-1}\pi_{nc}(a^n)e e^{-1}\pi_c(a^n)e \bar{\kappa}(a^n, x) = \tau(a^n x)\psi(a^n x)\sigma_{nc}(a^n)A^n\bar{\kappa}(a^n, x). \end{aligned}$$

We also get

$$d a^n(\bar{\tau}(x)) = d a^n(\tau(x) \psi(x)) = \tau(a^n x) \sigma(a^n) \psi(x).$$

Therefore we see that

$$\psi(a^n x) \sigma_{nc}(a)^n A^n \bar{\kappa}(a^n, x) = \sigma_c(a)^n \sigma_{nc}(a)^n \psi(x).$$

Let $B = \sigma_c(a)$. Since $\sigma_{nc}(a)$ commutes with A and $\bar{\kappa}$, we get

$$B^{-n} \psi(a^n x) A^n \bar{\kappa}(a^n, x) = \sigma_{nc}(a)^n \psi(x) \sigma_{nc}(a)^{-n}.$$

By Lusin's theorem, for every $\varepsilon > 0$, there exists a closed subset $M_\varepsilon \subset M$ such that the measure of M_ε is at least $1 - \varepsilon$ and ψ is continuous on M_ε . The transformation induced by a on M_ε is measure preserving. Hence, by Poincaré's recurrence theorem, for a.e. $x \in M_\varepsilon$ there are sequences of integers $n_k \rightarrow \infty$ and $m_k \rightarrow -\infty$ such that $a^{n_k} x, a^{m_k} x \in M_\varepsilon$, $a^{m_k} x \rightarrow x$ and $a^{n_k} x \rightarrow x$ as $k \rightarrow \infty$. Since A, B and the range of $\bar{\kappa}$ belong to compact groups, it follows that $\sigma_{nc}(a)^{m_k} \psi(x) \sigma(a)_{nc}^{-m_k}$ and also $\sigma(a)_{nc}^{n_k} \psi(x) \sigma_{nc}(a)^{-n_k}$ stay bounded as $k \rightarrow \infty$. As $\sigma_{nc}(a) \in GL(N, \mathbb{R})$ is diagonalizable, this is only possible if $\psi(x)$ commutes with $\sigma_{nc}(a)$. Thus we get that

$$\psi(a x) A \bar{\kappa}(a, x) = B \psi(x) \quad (*)$$

for a.e. $x \in M$ as M_ε has measure arbitrarily close to 1.

Now we pick the element a satisfying the weak mixing assumption and will show that this last cohomology equation forces ψ to be constant modulo a suitable compact subgroup. Let \mathcal{A} and \mathcal{B} denote the closure of the powers $A^n, n \in \mathbb{Z}$ and $B^n, n \in \mathbb{Z}$ respectively. Then both \mathcal{A} and \mathcal{B} are compact abelian groups. Let \mathcal{C} denote the closure of $(A^{-1}, B)^n, n \in \mathbb{Z}$ in $\mathcal{A} \times \mathcal{B}$. Consider the extension α of a on $M \times \mathcal{C}$ given by

$$\alpha(x, y, z) = (a x, A^{-1} y, B z).$$

Since (A^{-1}, B) generates a dense subgroup of \mathcal{C} , translation on \mathcal{C} by (A^{-1}, B) is ergodic w.r.t. Haar measure. Since a is weak mixing, it follows that the extension α is still ergodic w.r.t. the product of Haar measure on \mathcal{C} with the measure on M . As above, let $M_\varepsilon \subset M$ be a set of measure $1 - \varepsilon$ on which ψ is continuous. Since the transformation induced by α on $M_\varepsilon \times \mathcal{C}$ is ergodic, there is a point $(x, y, z) \in M_\varepsilon \times \mathcal{C}$ whose α -orbit intersected with $M_\varepsilon \times \mathcal{C}$ is dense in $M_\varepsilon \times \mathcal{C}$. Thus, given $x^* \in M_\varepsilon$, there is a sequence of integers n_k such that $a^{n_k} x \rightarrow x^*$, $a^{n_k} x \in M_\varepsilon$, $A^{-n_k} y \rightarrow y$ and $B^{n_k} z \rightarrow z$. It follows that $A^{-n_k} \rightarrow \text{Id}$ and $B^{n_k} \rightarrow \text{Id}$. Since ψ is continuous on M_ε and A commutes with the image of $\bar{\kappa}$, equation $(*)$ implies that

$$\psi(x^*) = \lim_{n_k \rightarrow \infty} \psi(a^{n_k} x) = \lim_{n_k \rightarrow \infty} B^{n_k} \psi(x) A^{-n_k} \bar{\kappa}(a^{n_k}, x)^{-1} = \psi(x) \lim_{n_k \rightarrow \infty} \bar{\kappa}(a^{n_k}, x)^{-1}.$$

Let C be the compact subgroup of $H \subset GL(N, \mathbb{R})$ which contains the image of $\bar{\kappa}$ and let $pr : \mathcal{G} \rightarrow \mathcal{G}/C$ denote the projection. Then $pr \circ \psi$ is constant a.e. on M_ε . Since the M_ε have measure arbitrarily close to 1, $pr \circ \psi$ is constant a.e. on M . Thus there is $\psi_0 \in \mathcal{G}$ such that for a.e. x , $\psi(x) \in \psi_0 C$. By the construction of M' as $\frac{P \times H^0 \setminus H}{H}$, the c frames $\bar{\tau}(x) = \tau(x) \psi_0$ belong to the fiber of P over $x \in M$. Hence this fiber is just the H -orbit of any of the c frames in $\bar{\tau}(x)$. In particular, we can choose the H -reduction P smoothly. Hence M' is a smooth finite manifold cover of M with a smooth Γ -action on M' . Pick a connected component M'_0 of M' . Then some subgroup Γ_0 of finite index of Γ preserves M'_0 . Passing to a finite power of a if necessary, we may assume that $a \in \Gamma_0$, and hence its lift to M' preserves M'_0 . Let \hat{a} denote this restriction.

By choice of a , \hat{a} is weak mixing and also an affine automorphism of the homogeneous space M'_0 . The superrigidity framing on M' restricts to a superrigidity framing for Γ_0 on M'_0 . Our previous argument for the multivalued function $\psi : M \rightarrow \mathcal{G}$ can be applied to the single valued function $M'_0 \rightarrow GL(N, \mathbb{R})$ determined by a superrigidity framing. Thus we get a constant translation τ' of the standard frame on M'_0 which differs from a superrigidity framing for the Γ_0 -action on M'_0 by translations by elements of C . Thus τ' is a superrigidity framing for the Γ_0 -action on M'_0 , a conjugate of the representation π and a cocycle taking values in C . \diamond

Now we are ready to prove the main theorem of this section.

Theorem 15 *Let G be a linear semisimple Lie group all of whose simple factors have real rank at least 2. Let Γ be an irreducible lattice in G . Then a sufficiently small C^1 -perturbation of an algebraic Anosov action of Γ on a nilmanifold M is C^∞ -conjugate to the original action, by a conjugacy C^1 -close to the identity.*

Proof: Let ρ be an algebraic Anosov action of Γ on a nilmanifold M of dimension N . Thus its universal cover \mathcal{N} is a nilpotent Lie group.

Step 1: Translation invariance of the superrigidity framing We will show that the hypotheses of Proposition 13 hold.

Let us first show that the restriction of the action to Δ is still Anosov. Fix a translation invariant framing τ on M . Let $\sigma : \Gamma \rightarrow SL(N, \mathbb{R})$ denote the linear part of the Γ -action, and let $\tilde{\sigma}$ denote the associated map into the adjoint group $PSL(N, \mathbb{R})$. Then $\rho(g)$ is an Anosov diffeomorphism precisely when $\tilde{\sigma}(g)$ does not have eigenvalues on the unit circle. As ρ is Anosov, this implies that $\tilde{\sigma}(\Gamma)$ is not a relatively compact subgroup of $PSL(N, \mathbb{R})$. By Margulis' superrigidity theorem, $\tilde{\sigma}$ extends to a homomorphism $G \rightarrow PSL(N, \mathbb{R})$ [29, Theorem 6.16].

Now suppose that $g \in \Gamma$ is an Anosov element. Let g_s be its semisimple Jordan component. Further decompose g_s as a product $g_s = kp$ of commuting elements k and p where all eigenvalues of k have modulus 1, and all eigenvalues of the polar part p are positive. Since G is an algebraic group, both g_s and p also belong to G . Since $\rho(g)$ is Anosov, $\tilde{\sigma}(g), \tilde{\sigma}(g_s)$ and also $\tilde{\sigma}(p)$ do not have eigenvalues on the unit circle. Let D be a Cartan subgroup of G containing Δ . Let A be the \mathbb{R} -split factor of D . Then p is conjugate to some element $g' \in A$. Pick δ in Δ such that the polar part of δ makes a sufficiently small angle with g' . Then $\sigma(\delta)$ also does not have eigenvalues on the unit circle. Hence $\rho(\delta)$ is Anosov.

Since some element $a \in \Delta$ is Anosov, $\rho(a)^k$ for all k is also Anosov and thus weak mixing on all connected finite covers of M . It remains to prove the local rigidity of the action for some Cartan subgroup Δ of Γ . We will indicate two arguments for proving that. One argument works in general and uses Corollary 4.

Semisimplicity of the linear part follows from semisimplicity of G and the fact that Δ is a Cartan subgroup. The only remaining condition is the absence of elements with roots of unity as eigenvalues in the abelianization. This easily follows from the following unpublished algebraic result of G. Prasad and A. Rapinchuk which strengthens a result of V. E. Voskresenskii [47].

Theorem (Prasad-Rapinchuk) *Let G and Γ be as before. Let $\mu : G \rightarrow GL(N, \mathbb{R})$ be a linear representation of G . Then there exists a Cartan subgroup Δ of Γ such that for all $\delta \neq 1 \in \Delta$, the sum of the multiplicities of roots of unity which are eigenvalues of $\mu(\delta)$ is the dimension of the 0-weight space of μ .*

Let μ be the representation of Γ on the abelianization $\frac{\mathfrak{h}}{[\mathfrak{h}, \mathfrak{h}]}$ of the Lie algebra \mathfrak{h} of N . Note that the kernel of μ is finite by Margulis' finiteness theorem [29, Theorem 4']. Hence π is essentially faithful. By the last proposition and the presence of an Anosov element, we can pick a Cartan $\Delta \subset \Gamma$ such that $\mu(\lambda)$ does not have any roots of unity as eigenvalues for any $\lambda \in \Delta \neq 1$. Then Corollary 4 applies to give a C^∞ -conjugacy between $\rho|_\Delta$ and the perturbation $\rho^*|_\Delta$.

If Γ is a cocompact lattice in G , there is an alternate argument. Suppose Γ acts on a nilmanifold M by ρ , and that $\tilde{\rho}$ is a C^1 -small perturbation of ρ . Induce ρ and $\tilde{\rho}$ to G actions on $G \times_\Gamma M$. Restrict these actions to a maximal split Cartan A . Let K denote the compact part of the centralizer of A in G . Then both actions descend to C^1 -close actions σ and $\tilde{\sigma}$ on $K \backslash G \times_\Gamma M$. In fact, σ is a twisted Weyl chamber flow. By Corollary 5, there is a conjugacy ϕ close to the identity and an automorphism α of A close to the identity such that $\sigma = \phi \circ \tilde{\sigma} \circ \alpha$. Note that the fiber M over $K1\Gamma$ is invariant under $\Delta := A \cap \Gamma$ (where we assume that Γ intersects A in a lattice after a conjugation and moving the base point if necessary). Hence $\phi(M)$ is invariant under $\alpha^{-1}(\Delta)$. Note that $\phi(M)$ is close to M . Hence the projection X of $\phi(M)$ to $K \backslash G/\Gamma$ is an $\alpha^{-1}(\Delta)$ -invariant compact set contained in an ε -neighborhood of $K1\Gamma$. By the standard hyperbolic arguments, this set has to be contained in the neutral manifold of the A action on $K \backslash G/\Gamma$. If α is not the identity, there is $\delta \in \Delta$ such that $\alpha^{-1}(\delta)\delta^{-1}$ moves $K1\Gamma$ by more than 2ε . Then $\alpha^{-1}(\delta)$ has to move X outside the ε -neighborhood of $K1\Gamma$ which is impossible.

Thus $\alpha = id$ and ϕ is a conjugacy between $\tilde{\sigma}$ and σ . Since X is contained in the A -orbit of $K1\Gamma$, ϕ is a conjugacy of the suspension of the ρ and $\tilde{\rho}$ -actions to \mathbb{R}^k -actions. This implies that ρ and $\tilde{\rho}$ are smoothly conjugate.

Thus all the conditions of Proposition 13 have been verified, and we get a translation invariant superrigidity framing τ^* for a subgroup Γ_0 of finite index in Γ . We will also assume henceforth that ρ and $\tilde{\rho}$ agree on Δ . Since τ^* is a constant translate of τ by some element $d \in GL(N, \mathbb{R})$, we may further assume that $\tau = \tau^*$, conjugating the relevant superrigidity representation and cocycle by d , if necessary.

Step 2: Local C^∞ rigidity on a subgroup of finite index

By a result of S. Hurder, there is a subgroup Γ^* of finite index in Γ which fixes a point o in M' under the unperturbed action ρ [12, Corollary 2]. G. A. Margulis showed that the first cohomology of Γ^* with coefficients in any finite dimensional representation of Γ^* vanishes [29, Theorem 3']. Hence, by a theorem of D. Stowe, a sufficiently close perturbed action $\tilde{\rho}$ still has a fixed point o' for Γ^* [46]. Since ρ and $\tilde{\rho}$ coincide on Δ , it follows that $\Delta \cap \Gamma^*$ fixes both o and o' . Since $\Delta \cap \Gamma^+$ contains Anosov elements and o and o' are close, this forces $o = o'$. Hence we may assume that Γ^* fixes o .

Consider the normal subgroup $\Gamma_A \subset \Gamma^*$ which is generated by Anosov elements of Γ^* with respect to ρ . By Margulis' finiteness theorem, Γ_A has finite index in Γ [29, Ch. IX, Theorem 5.4]. Since Γ_A has Kazhdan's property (T), Γ_A has a finite set F of generators such that $\rho(\gamma)$ is Anosov for all $\gamma \in F$ (this follows easily from the standard proof of finite generation of Kazhdan groups [50, Theorem 7.1.5]).

Write $M = \mathcal{N}/\Lambda$ as a quotient of the universal cover \mathcal{N} of M . Fix γ in F , and lift $\rho(\gamma)$ and $\tilde{\rho}(\gamma)$ to diffeomorphisms g and \tilde{g} on the universal cover \mathcal{N} of M such that both g and \tilde{g} fix a point $p \in \mathcal{N}$ covering the common fixed point $o \in M$. Then g and \tilde{g} are C^1 -close on \mathcal{N} . Denote the lift of τ to \mathcal{N} again by τ . Then g transform τ by $\sigma(\gamma)$. By Step 1, \tilde{g} transforms $\tau(x)$ by $\pi(\gamma) \circ \kappa(\gamma, x)$. Since κ takes values in a compact group C commuting with the image of π , the stable and unstable distributions of \tilde{g} (with respect to a right invariant metric on \mathcal{N}) are determined by the eigenspaces of $\pi(\gamma)$, and hence are invariant under right translation by \mathcal{N} . Thus the stable foliation of \tilde{g} (resp.

g) is an orbit foliation of some subgroup \tilde{L} (resp. L) of \mathcal{N} . The following lemma is presumably well known. For completeness we include an elegant proof suggested to us by G. Prasad.

Lemma 16 *Let \mathcal{N} be a simply connected nilpotent group, and H, L two closed connected subgroups of \mathcal{N} . Suppose that L is contained in a tubular neighborhood of H of fixed size α with respect to a right invariant Riemannian metric on N . Then L is a subset of H .*

Proof: Recall that \mathcal{N} can be embedded into the upper triangular matrices. Thus \mathcal{N}, H and L are unipotent algebraic groups. Then the homogeneous space $H \setminus \mathcal{N}$ is an affine variety [3, Corollary 6.9]. Since L is a unipotent group, the L -orbit of $H \cdot 1$ in $H \setminus \mathcal{N}$ is closed by Kolchin's theorem [3, Proposition 4.10]. Hence this orbit is homeomorphic with $(H \cap L) \setminus L$. Now suppose $L \subset HB$ where $B \subset \mathcal{N}$ is compact. Then the L -orbit of $H \cdot 1$ in $H \setminus \mathcal{N}$ is bounded, and thus compact. Since H is unipotent, again $(H \cap L) \setminus L$ is an affine variety, and thus cannot be compact. \diamond

We will now show that the stable foliations for g and \tilde{g} coincide, or equivalently that $H = L$. Let i be the injectivity radius of M . We will assume that $\tilde{\rho}$ is so close to ρ that the unique topological conjugacy ϕ_γ between $\rho(\gamma)$ and $\tilde{\rho}(\gamma)$ is $i/100$ -close to the identity. Note that ϕ_γ takes stable manifolds of $\rho(\gamma)$ to stable manifolds of $\tilde{\rho}(\gamma)$. Since the fixed points of $\rho(\gamma)$ are isolated, we may further assume that $\phi_\gamma(o) = o$. Let $\bar{\phi}_\gamma$ be the lift of ϕ_γ to \mathcal{N} such that $\bar{\phi}_\gamma(p) = p$. Then the distance between $\bar{\phi}_\gamma$ and the identity is again at most $i/100$. Moreover, $\bar{\phi}_\gamma$ takes the stable manifold Lp to $\tilde{L}p$. Thus the Hausdorff distance between L and \tilde{L} is at most $i/100$. Lemma 16 now shows that $L = \tilde{L}$.

Thus we now know that g and \tilde{g} have the same stable and also unstable manifolds $\mathcal{W}^s(x) = Lx$ and $\mathcal{W}^u(x)$. As before, \mathcal{W}^u is the orbit foliation of a subgroup, say L^u of \mathcal{N} . Set $f = g^{-1}\tilde{g}$. Then f preserves \mathcal{W}^s and \mathcal{W}^u . Since $\rho(\gamma)$ and $\tilde{\rho}(\gamma)$ are C^1 -close, their induced maps on $\Lambda = \pi_1(M)$ coincide. Hence f commutes with the action of Λ on N , and fixes p as well as all $p\lambda$ for $\lambda \in \Lambda$.

Now note that $\mathcal{W}^u(p) \cap \mathcal{W}^s(p\lambda)$ consists of at most one point. Otherwise, $L \cap L^u \neq 1$, and hence contains a one-parameter subgroup as the exponential map of \mathcal{N} is a diffeomorphism. That however contradicts the transversality of $\mathcal{W}^s(p) = Lp$ and $\mathcal{W}^u(p) = L^up$.

Since f fixes all $p\lambda$, it follows that f also fixes any intersection point $\mathcal{W}^s(p\lambda) \cap \mathcal{W}^u(p)$. Since $\rho(\gamma)$ is Anosov, $\mathcal{W}^s(o)$ is dense in M . Hence the set of intersection points $\mathcal{W}^s(p\lambda) \cap \mathcal{W}^u(p)$ is dense in $\mathcal{W}^u(p)$. Thus f fixes $\mathcal{W}^u(p)$. As $\mathcal{W}^u(p)$ is dense in M , $f = id$, and hence $\rho(\gamma) = \tilde{\rho}(\gamma)$.

Step 3: Local C^∞ -rigidity

By Step 2, we may assume that ρ and $\tilde{\rho}$ coincide on Γ_A . Now consider the normal subgroup $\Gamma_c \subset \Gamma$ which is generated by all Anosov elements of Γ with respect to ρ . By Margulis' finiteness theorem, Γ_c has finite index in Γ . Let $\gamma \in \Gamma_c$ be an Anosov element for ρ . Then some finite power γ^k belongs to Γ_A . Since both $\rho(\gamma)$ and $\tilde{\rho}(\gamma)$ are close and commute with $\rho(\gamma)^k = \tilde{\rho}(\gamma)^k$, $\rho(\gamma)$ and $\tilde{\rho}(\gamma)$ coincide since the centralizer of an Anosov element is discrete by expansiveness.

To finish the proof of the proposition, we will show that ρ and $\tilde{\rho}$ automatically coincide on Γ . Let τ be the projection of a right-invariant framing as above. Fix $f \in \Gamma$, and set $\tau^f(x) = Df(\tau(f^{-1}(x)))$. Since $\Gamma_c \subset \Gamma$ is normal, one easily sees that τ^f transforms under Γ_c by the representation σ conjugated by the element $\sigma(f)$

$$Dn\tau^f(x) = \tau^f(nx)\sigma(f)^{-1}\sigma(n)\sigma(f).$$

Thus the transformation law for τ^f under Γ_c is completely given by a homomorphism. In particular, the cocycle κ taking values in some compact group is trivial. Applying the argument of Step 2

to τ^f , one sees that τ^f is a constant translate of τ . This means that the derivative Df in terms of the framing τ is constant. Thus we get a homomorphism $\sigma^\# : \Gamma \rightarrow GL(N, \mathbb{R})$ where N is the dimension of M .

We will now finish the proof of the theorem by showing that σ and $\sigma^\#$ coincide on Γ . The following argument for this is due to G. A. Margulis, and relies on his structure theorem for homomorphisms of arithmetic groups in higher rank semisimple groups into algebraic groups [29, Ch. VIII, Theorem 3.12]. Suppose that Γ is an arithmetic group defined over a number field K . By Margulis' theorem, there is a homomorphism $\nu : \Gamma \rightarrow GL(N, \mathbb{R})$ with finite image and a rational homomorphism ϕ defined on the \mathbb{Q} -group obtained from G by restriction of scalars from K to \mathbb{Q} such that $\sigma^\#(\gamma) = \nu(\gamma)\phi(\gamma)$. Since ϕ and σ are rational and coincide on the intersection of the kernel of ν with Γ_A , ϕ and σ coincide. Since $\sigma^\#$ and σ coincide on Γ_c , ν is trivial on Γ_c . Hence ν is defined on the finite group Γ/Γ_c . On the other hand, ρ^* is C^1 -close to ρ . Hence ν is arbitrarily close to the identity. This is impossible as the trivial representation is isolated within the space of representations of a fixed finite group. \diamond

4 Lattice actions on boundaries

In this section we will demonstrate the local C^∞ -rigidity of projective actions of cocompact lattices in semisimple groups of the noncompact type.

Theorem 17 *Let G be a connected semisimple Lie group with finite center and without compact factors. Suppose that the real rank of G is at least 2. Let $\Gamma \subset G$ be a cocompact irreducible lattice and P a parabolic subgroup of G . Then the action of Γ on G/P by left translations is locally C^∞ -rigid.*

The main idea in our proof is to reduce this rigidity problem to the transversal rigidity of the orbit foliation of a suitable normally hyperbolic action on $\Gamma \backslash G$. We would like to thank E. Ghys for suggesting this approach. In fact, Ghys used a similar duality to obtain the smooth classification of boundary actions of Fuchsian groups [7].

In the special case of a projective action of a cocompact lattice in $SL(n+1, \mathbb{R})$ on the n -sphere S^n , M. Kanai established a local rigidity result for small perturbations in the C^4 -topology for all $n \geq 21$ [13]. His method is completely different and relies on the vanishing of a certain cohomology.

We begin with a brief review of foliated bundles and the suspension construction (cf. [10] and also [49] for a more detailed description). Let M be a compact manifold with cover \tilde{M} (let us emphasize that \tilde{M} need not be the universal cover). Let Γ be the group of covering transformations. Given a manifold V and a homomorphism $\rho : \Gamma \rightarrow \text{Diff}(V)$, we can form the manifold

$$E = (\tilde{M} \times V)/\Gamma$$

where Γ acts on $\tilde{M} \times V$ by the diagonal action $\gamma(p, v) = (\gamma p, \rho(\gamma)v)$. Then E becomes a fiber bundle over M with fiber V by projecting to the first factor. Note that the manifolds $\tilde{M} \times \{v\}$, $v \in V$, project to a foliation \mathcal{F} of E which is transverse to the fibers and has complementary dimension. This is the so-called *suspension construction*. Any bundle with a transversal foliation \mathcal{F} of complementary dimension is called a *foliated bundle*. One can show that any such foliated bundle arises via the suspension construction where the homomorphism $\rho : \Gamma \rightarrow \text{Diff}(V)$ is the *holonomy homomorphism* of the foliation defined as follows. Let $p \in M$, and identify V with the

fiber V_p . If c is a loop at p , then for $v \in V$, let $c(v)$ be the beginning point of a horizontal lift of c with endpoint v , i.e. a lift which is tangent to \mathcal{F} . This defines a diffeomorphism of V which only depends on the homotopy class of c .

Suppose now that V is compact and that ρ' is another action of Γ on V . If ρ' is sufficiently C^1 -close to ρ (on some finite generating set of $\pi_1(M)$), then the bundle E' obtained from ρ' via the suspension construction is diffeomorphic to E via a bundle map. Moreover, the pullback of the natural transverse foliation \mathcal{F}' from E' to E is C^1 -close to \mathcal{F} . We will henceforth identify E' with E , and \mathcal{F}' with its pull-back. Finally, ρ and ρ' are conjugate by a C^∞ diffeomorphism of V precisely when there exists a C^∞ diffeomorphism of E which carries \mathcal{F} to \mathcal{F}' .

Let G be a connected semisimple Lie group with finite center and without compact factors. Let $\Gamma \subset G$ be an irreducible lattice. We will now exhibit the action of Γ on a boundary G/P of G , P a parabolic subgroup, as a holonomy action of a suitable “stable” foliation. We refer to [48, 50] for details on parabolic subgroups and boundaries of G .

Let $P = LC U^+$ be a Langlands decomposition of P (unique up to conjugacy). Here U^+ is the unipotent radical of P , and LC is a Levi subgroup of P , i.e. a maximal reductive subgroup (i.e. a product of a semisimple and an abelian group). Moreover, C is the intersection of the Levi subgroup LC with a suitable split Cartan A , and L is reductive and commutes with C . If P is a minimal parabolic then U^+ is a maximal unipotent subgroup of G , C is a split Cartan, and L is just the maximal compact subgroup of the centralizer of A in G .

Let $c \in C$ be a regular element of C , i.e. an element such that all roots of G which are not zero on the Lie algebra \mathfrak{c} of C are also not zero on $\log c$. We may further assume that c lies on the boundary of a positive Weyl chamber of A . The Lie algebra of U^+ is spanned by the rootspaces of G whose roots are positive on \mathfrak{c} . Hence these are precisely the roots which are positive on $\log c$. Moreover, the Lie algebra of P is the sum of the Lie algebra of the centralizer of A and the rootspaces which are nonnegative on $\log c$. Let P act on $\Gamma \backslash G$ by right translations. Then the orbit foliation of P is precisely the weak stable foliation of c . Let $G = KAN^+$ be an Iwasawa decomposition of G , and set $M_P = K \cap P$. Then M_P is the centralizer of c in K . Note that $\Gamma \backslash G/M_P \rightarrow \Gamma \backslash G/K$ is a bundle with fiber K/M_P , and that the projection \mathcal{W}^+ of the orbit foliation of P on $\Gamma \backslash G$ to $\Gamma \backslash G/M_P$ is a transverse foliation of complementary dimension since $KP = G$. As C commutes with M_P , the right action of C descends from $\Gamma \backslash G$ to $\Gamma \backslash G/M_P$ with \mathcal{W}^+ as the weak unstable foliation of c .

Let us next describe the holonomy representation of \mathcal{F} with respect to Γ and the cover G/M_P of $\Gamma \backslash G/M_P$. The fiber K/M_P of the bundle $\Gamma \backslash G/M_P \rightarrow \Gamma \backslash G/K$ is diffeomorphic with G/P , again since $G = KP$. Henceforth, we will identify the fiber with G/P .

Lemma 18 *The holonomy representation ρ of the \mathcal{W}^+ -foliation on the fiber G/P is given by right multiplication by γ on G/P for $\gamma \in \Gamma$.*

Proof: Consider the cover G/K of $\Gamma \backslash G/K$ and the corresponding fiber bundle $G/M_P \rightarrow G/K$. The K/M_P -fiber of $1M_P$ gets identified with that of γM_P by left multiplication by γ . Then $\rho(\gamma)(kM_P)$ is that lM_P such that $lP \supset \gamma kM_P$. Identifying K/M_P with G/P , this simply means that $lP = \gamma kP$.

One can also see this more geometrically by identifying the orbit foliation of P with the weak unstable foliation of c . Identify the fiber K/M_P first with a suitable subspace of the unit tangent sphere of $1K$, and second with a subspace of the sphere at infinity of the globally symmetric space G/K by projecting the unit tangent vector v to the asymptote class $v(-\infty)$ of the geodesic ray $-v$ defines. One can check that the image is the homogeneous space G/P . Again, v gets identified with $d\gamma(v) \in T_{\gamma K}G/K$. As $d\gamma(v)(-\infty) = \gamma x$, the holonomy image $\rho(\gamma)(v)$ is that $w \in T_{1K}G/K$

with $w(\infty) = \gamma x$. Under our identification of $T_1 K G/K$ with G/P , this just means $\rho(\gamma)(x) = \gamma x$.
 \diamond

Proof of Theorem 17: Let us first describe the weak stable foliation \mathcal{W}^- of c and its holonomy. Let U^- be the unipotent subgroup of G whose Lie algebra is spanned by the rootspaces of G whose roots are negative on c or equivalently \mathfrak{c} . Let $P^- = LCU^+$ be the opposite parabolic subgroup. Then K/M_P can be identified with G/P^- . Geometrically, viewing K/M_P with a subset of $T_1 G/K$, this identification is given by sending a suitable unit tangent vector v to $v(-\infty)$. This yields a diffeomorphism $\theta : G/P \rightarrow G/P^-$. Now the holonomy representation ρ^- of \mathcal{W}^- is given by $\rho^-(\gamma)(\theta(x)) = \theta(\rho(\gamma)(x))$.

Now a C^1 -small perturbation $\tilde{\rho}$ of ρ gives a C^1 -small perturbation $\tilde{\rho}^-$ of ρ^- . These perturbations of the holonomies define C^1 -small perturbations $\tilde{\mathcal{W}}^+$ and $\tilde{\mathcal{W}}^-$ of the foliations \mathcal{W}^+ and \mathcal{W}^- . Note that the orbit foliation \mathcal{O} of the Levi subgroup $H = LC$ of P on $\Gamma \backslash G$ is invariant under the action of M_P , and hence descends to a foliation \mathcal{R} on $\Gamma \backslash G/M_P$. Moreover, \mathcal{R} is the leafwise intersection of \mathcal{W}^+ and \mathcal{W}^- , and this intersection is transversal. Hence the intersection of $\tilde{\mathcal{W}}^+$ and $\tilde{\mathcal{W}}^-$ is a C^1 -small perturbation $\tilde{\mathcal{R}}$ of \mathcal{R} . By Corollary 7, there is a smooth orbit equivalence Φ from $\tilde{\mathcal{R}}$ to \mathcal{R} that Φ is C^1 -close to the identity. Hence the pull-back foliations $\Phi^{-1}(\tilde{\mathcal{W}}^+)$ and $\Phi^{-1}(\tilde{\mathcal{W}}^-)$ are C^1 -close to \mathcal{W}^+ and \mathcal{W}^- , and are also saturated under the \mathcal{R} foliation. In particular, they are invariant under the action of c . However, it is clear that \mathcal{W}^+ and \mathcal{W}^- are the unique c -invariant foliations C^1 -close to \mathcal{W}^+ and \mathcal{W}^- . Indeed, any vector close to the tangent space of \mathcal{W}^- but not tangent to \mathcal{W}^- has a non-trivial strong unstable component, and eventually gets expanded by c under forward iteration. Thus eventually, the unstable component would dominate, and the foliation could not be C^1 -close to \mathcal{W}^- .

We conclude that Φ is a C^∞ -orbit equivalence between $\tilde{\mathcal{W}}^+$ and \mathcal{W}^+ . Hence the holonomy actions $\tilde{\rho}$ and ρ are C^∞ -conjugate.
 \diamond

References

- [1] D.V. Anosov, *Roughness of geodesic flows on compact manifolds of negative curvature* (in Russian), Dokl. Akad. Nauk SSSR, **145** (1962), 707–709.
- [2] D.V. Anosov, *Geodesic Flows on closed Riemannian manifolds of negative curvature*, Ppoc. Steklov Math. Inst **90** (1967).
- [3] A. Borel, *Linear Algebraic Groups*, 2nd ed., Springer Verlag, New York 1991.
- [4] J. Brezin and C. C. Moore, *Flows on homogeneous spaces: a new look*, Amer. J. of Math. **103** (1981), 571–613.
- [5] K. Burns and R. J. Spatzier, *On topological Tits buildings and their classification*, Publ. Math. IHES **65** (1987), 1–35.
- [6] K.-T. Chen, *Equivalence and decomposition of vector fields about an elementary critical point*, Amer. J. of Math. **85** (1963), 639–722.
- [7] E. Ghys, *Rigidité différentiables des groupes fuchsien*, Publ. Math. IHES **78** (1993), 163–185.
- [8] E. Goetze and R. J. Spatzier, *Smooth classification of Cartan actions of higher rank semisimple Lie groups and their lattices*, preprint.
- [9] M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics **583**, Springer Verlag, Berlin 1977.
- [10] M. Hirsch and W. P. Thurston, *Foliated bundles, invariant measures and flat manifolds*, Ann. of Math. **101** (1975), 369–390.
- [11] S. Hurder, *Rigidity for Anosov actions of higher rank lattices*, Ann. Math. **135** (1992), 361–410.

- [12] S. Hurder, *Affine Anosov actions*, Mich. Math. J. **40** (1993), 561–575.
- [13] M. Kanai, *A new approach to the rigidity of discrete group actions*, preprint 1995.
- [14] A. Katok, *Hyperbolic measures for actions of higher-rank abelian groups*, preprint 1996.
- [15] A. Katok, *Normal forms and invariant geometric structures on transverse contracting foliations*, preprint 1996.
- [16] A. Katok, *Local properties of hyperbolic sets* (in Russian), Supplement to the Russian translation of “Differentiable Dynamics” by Z. Nitecki, Mir, Moscow, 1975, 214–232.
- [17] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
- [18] A. Katok and J. Lewis, *Local rigidity for certain groups of toral automorphisms*, Israel J. of Math. **75** (1991), 203–241.
- [19] A. Katok and J. Lewis, *Global rigidity for lattice actions on tori and new examples of volume preserving actions*, Israel J. of Math. **93** (1996), 253–281.
- [20] A. Katok, J. Lewis and R. J. Zimmer, *Cocycle superrigidity and rigidity for lattice actions on tori*, Topology **35** (1996), 27–38.
- [21] A. Katok and R. J. Spatzier, *First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity*, Inst. Hautes Études Sci. Publ. Math. No. **79** (1994), 131–156.
- [22] A. Katok and R. J. Spatzier, *Subelliptic estimates of polynomial differential operators and applications to rigidity of abelian action*, Math. Res. Lett. **1** (1994), no. 2, 193–202.
- [23] A. Katok and R. J. Spatzier, *Invariant Measures for Higher Rank Hyperbolic Abelian Actions*, Erg. Th. and Dynam. Syst. **16** (1996), no. 4, 751–778.
- [24] A. Katok and R. J. Spatzier, *Differential rigidity of hyperbolic abelian actions*, preprint 1992.
- [25] A. Katok and R. J. Spatzier, *Non-stationary normal forms and rigidity of group actions*, Elect. Res. Ann. of the AMS, **2** (1996), no. 3, 124–133.
- [26] E. Leuzinger, *New geometric examples of Anosov actions*, Ann. Global Anal. Geom. **12** (1994), 173–181.
- [27] J. Lewis, *The algebraic hull of the derivative cocycle*, preprint (1993).
- [28] R. Mañé, *A proof of the C^1 stability conjecture*, Inst. Hautes Études Sci. Publ. Math. No. **66** (1987), 161–210.
- [29] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer Verlag, Berlin 1991.
- [30] D. Montgomery and L. Zippin, *Topological transformation groups*, Interscience Publishers, Inc., New York, 1955.
- [31] W. Parry, *Ergodic properties of affine transformations and flows on nilmanifolds*, Amer. J. Math. **91** (1969), 757–771.
- [32] G. Prasad and M. Raghunathan, *Cartan subgroups and lattices in semisimple groups*, Ann. of Math. **96** (1972), 296–317.
- [33] C. Pugh and M. Shub, *Ergodicity of Anosov actions*, Invent. Math. **15** (1972), 1–23.
- [34] N. Qian, *Anosov automorphisms for nilmanifolds and rigidity of group actions*, Erg. Th. and Dynam. Syst. **15** (1995), 341–359.
- [35] N. Qian, *Topological deformation rigidity of actions of higher rank lattices*, Math. Res. Lett. **1** (1994), 485–499.
- [36] N. Qian, *Tangential flatness and global rigidity of higher rank lattice actions*, preprint.
- [37] N. Qian, *Smooth conjugacy for Anosov diffeomorphisms and rigidity of Anosov actions of higher rank lattices*, preprint.
- [38] N. Qian and C. Yue, *Local rigidity of Anosov higher rank lattice actions*, preprint 1996.
- [39] N. Qian and R. J. Zimmer, *Entropy rigidity for semisimple group actions*, preprint.
- [40] M. Raghunathan, *Discrete subgroups of Lie groups*, Springer, New York 1972.
- [41] J. Robbin, *A structural stability theorem*, Ann. Math **94** (1971), 447–493.
- [42] R. Sacksteder, *Semigroups of expanding maps*, Trans. Amer. Math. Soc. **221** (1976), 281–288.
- [43] G. Seydoux, *Rigidity of ergodic volume-preserving actions of semisimple groups of higher rank on compact manifolds*, Trans. Amer. Math. Soc. **345** (1994), no. 2, 753–776.
- [44] S. Smale *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

- [45] S. Sternberg, *Local contractions and a theorem of Poincaré*, Amer. J. of Math. **79** (1957), 809–824.
- [46] D. Stowe, *The stationary set of a group action*, Proc. AMS **79** (1980), 139–146.
- [47] V. E. Voskresenskii, *Maximal tori without effect in semisimple algebraic groups*, (Russian) Mat. Zametki **44** (1988), no. 3, 309–318.
- [48] G. Warner, *Harmonic analysis on semi-simple Lie groups I*, Springer Verlag, Berlin 1972.
- [49] C. Yue, *Smooth rigidity of rank 1 lattice actions on the sphere at infinity*, Math. Res. Lett. **2** (1995), 327–338.
- [50] R. J. Zimmer, *Ergodic Theory and Semisimple Groups*, Boston, Birkhäuser, 1984.
- [51] R. J. Zimmer, *On the algebraic hull of an automorphism group of a principal bundle*, Comment. Math. Helv. **65** (1990), 375–387.

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